Self-Dual Higher Gauge Theory

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Outline

- Introduction And Motivation
- Chiral Fields In 6*d* And Their Twistorial Interpretation
- Non-Abelian Extensions And Supersymmetry
- Generalisations
- Conclusions And Outlook

Introduction And Motivation

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One of the big challenges in M-theory is the formulation of the so-called $\mathcal{N} = (2,0)$ theory. This a chiral superconformal gauge theory in six dimensions with maximal $\mathcal{N} = (2,0)$ supersymmetry.

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- a potential 2-form B with curvature 3-form $H = \mathrm{d}B$ such that $H = \star_6 H$,
- five scalars ϕ^{IJ} such that $\Box\phi^{IJ} = 0$, and
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Problem: How can this be promoted to an interacting non-Ablian theory?

Proposal: Combine twistor theory and categorified principal bundles.

- The central objects are double fibrations of the form

$$\begin{array}{ccc} & F & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ P & & M \end{array}$$

  where M, F and P are complex manifolds:
  - M space-time
  - F correspondence space
  - P twistor space

Twistor Basics

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  where M, F and P are complex manifolds:
  - M space-time
  - F correspondence space
  - P twistor space

- Then we have a correspondence between P and M, i.e. between points in one space and subspaces of the other:

$$\begin{array}{ccc} \pi_1(\pi_2^{-1}(x)) \hookrightarrow P & \leftrightarrow & x \in M \\ p \in P & \leftrightarrow & \pi_2(\pi_1^{-1}(p)) \hookrightarrow M \end{array}$$

Twistor Correspondence: $P \stackrel{\pi_1}{\leftarrow} F \stackrel{\pi_2}{\rightarrow} M$

- Using this correspondence, we can transfer data given on P to data on M and vice versa (e.g. vector bundles, sheaf cohomology groups, contact forms, ...).

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- Take some analytic object Ob_P on P and transform it to an object Ob_M on M; this in turn is constrained by some PDEs as $\pi_1^*\mathrm{Ob}_P$ has to be constant up the fibres of $\pi_1 : F \to P$.

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- Take some analytic object Ob_P on P and transform it to an object Ob_M on M; this in turn is constrained by some PDEs as $\pi_1^*\mathrm{Ob}_P$ has to be constant up the fibres of $\pi_1 : F \to P$.
- Under suitable topological conditions, the maps

$$\mathrm{Ob}_P \mapsto \mathrm{Ob}_M \qquad \text{and} \qquad \mathrm{Ob}_M \mapsto \mathrm{Ob}_P$$

define a bijection between $[\mathrm{Ob}_P]$ and $[\mathrm{Ob}_M]$ (the objects in question will only be defined up to equivalence).

Example: Penrose Transform

Consider 4d flat space $M = \mathbb{C}^4$ with $TM \cong S \otimes \tilde{S}$:

$$\begin{array}{ccc} & F = \mathbb{P}(\tilde{S}^\vee) = \mathbb{C}^4 \times \mathbb{P}^1 & \\ & \pi_1 \swarrow \quad \searrow \pi_2 & \\ P = \mathbb{P}^3 \setminus \mathbb{P}^1 & & M = \mathbb{C}^4 \end{array}$$

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Then,

$$H^1(P, \mathcal{O}_P(-2h-2)) \cong \left\{ \begin{array}{c} \text{zero-rest-mass fields} \\ \text{of helicity } h \text{ on } M \end{array} \right\}$$

Consider again $M = \mathbb{C}^4$ with $TM \cong S \otimes \tilde{S}$:

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Example: Penrose–Ward Transform

Consider again $M = \mathbb{C}^4$ with $TM \cong S \otimes \tilde{S}$:

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$$\boxed{\pi_1(\pi_2^{-1}(x)) = \mathbb{P}^1_x} \hookrightarrow P = \mathbb{P}^3 \setminus \mathbb{P}^1 \xleftarrow{\pi_1} F \xrightarrow{\pi_2} M = \mathbb{C}^4 \hookleftarrow \boxed{\mathbb{C}^2_p = \pi_2(\pi_1^{-1}(p))}$$

We have a natural bijection between equivalence classes of

- holomorphic M-trivial principal G-bundles over P and
- solutions to $F = \star_4 F$ on M with $F = \mathrm{d}A + \frac{1}{2}[A, A]$ and $A \in \Omega^1 \otimes \mathfrak{g}$.

Chiral Fields In 6*d* And Their Twistorial Interpretation

1111.2539 (JMP) with C Sämann

see also 1111.2585 (JGP) by Mason, Reid-Edwards & Taghavi-Chabert

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- Then choose coordinates $x^{AB} = -x^{BA}$ with $\partial_{AB} = -\partial_{BA}$, where $A, B, \ldots = 1, \ldots, 4$ and the metric is $\frac{1}{2}\varepsilon_{ABCD}$.
- Null-momentum p_{AB} is given by

$$\tfrac{1}{2} p_{AB} p_{CD} \varepsilon^{ABCD} = p_{AB} p^{AB} = 0$$

so that

$$p_{AB} \; = \; k_{Aa} k_{Bb} \varepsilon^{ab} \; , \qquad p^{AB} \; = \; \tilde{k}^{A\dot{a}} \tilde{k}^{B\dot{b}} \varepsilon_{\dot{a}\dot{b}} \; ,$$

where $a, \dot{a}, \ldots$ are $SL(2,\mathbb{C}) \times \widetilde{SL(2,\mathbb{C})}$ little group indices.

Chiral Fields

- Interested in fields that transform trivially under $\widetilde{SL(2,\mathbb{C})} \to$ chiral fields with $(\mathbf{2h+1},\mathbf{1})$ and $h \in \frac{1}{2}\mathbb{N}_0$ which we call spin

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- The $\mathcal{N}=(2,0)$ tensor multiplet consists of a self-dual 3-form $H = \mathrm{d}B$ in the $(\mathbf{3},\mathbf{1})$ representation, four Weyl fermions ψ^I in the $(\mathbf{2},\mathbf{1})$ and five scalars ϕ^{IJ} in the $(\mathbf{1},\mathbf{1})$:

$$\partial^{AC} H_{CB} \;=\; \partial^{AC}\psi_C \;=\; \Box\phi \;=\; 0\;,$$

where

$$\left\{\begin{array}{rcl} H &=& \mathrm{d}B \\ H &=& *H \end{array}\right\} \leftrightarrow \left\{\begin{array}{rcl} (H_{AB}, H^{AB}) &=& (\partial_{C(A}B_{B)}{}^C, \partial^{C(A}B_C{}^{B)}) \\ H^{AB} &=& 0 \end{array}\right\}$$

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- The corresponding plane waves are

$$H_{AB\,ab} \;=\; k_{A(a}k_{Bb)}\,\mathrm{e}^{\mathrm{i}\,x\cdot p}\;,\quad \psi_{Aa} \;=\; k_{Aa}\,\mathrm{e}^{\mathrm{i}\,x\cdot p}\;,\quad \phi \;=\; \mathrm{e}^{\mathrm{i}\,x\cdot p}\;.$$

Twistor Space For Chiral Fields

- Starting from space-time M with coordinates x^{AB}, define the correspondence space F to be $F := \mathbb{P}(S^\vee) \cong \mathbb{C}^6 \times \mathbb{P}^3$ with coordinates (x^{AB}, λ_A).

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- One can show that

$$P \;\cong\; T^\vee\mathbb{P}^3 \otimes \mathcal{O}_{\mathbb{P}^3}(2) \;\hookrightarrow\; \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathbb{C}^4 \;\cong\; \mathbb{P}^7 \setminus \mathbb{P}^3$$

so we may use coordinates (z^A, λ_A) with $z^A \lambda_A = 0$ and thus

$$\begin{array}{ccc} & F & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ P & & M \end{array}$$

with π_2 being the trivial projection and

$$\pi_1 \;:\; (x^{AB}, \lambda_A) \;\mapsto\; (z^A, \lambda_A) \;=\; (x^{AB}\lambda_B, \lambda_A).$$

Twistor Space For Chiral Fields

$$\begin{array}{ccc} & F & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ P & & M \end{array}$$

Because of $z^A = x^{AB}\lambda_B$ we have a geometric correspondence:

$$\begin{array}{ccc} \pi_1(\pi_2^{-1}(x)) \cong \mathbb{P}^3_x \hookrightarrow P & \leftrightarrow & x \in M \\ p \in P & \leftrightarrow & \pi_2(\pi_1^{-1}(p)) \cong \mathbb{C}^3_z \hookrightarrow M \end{array}$$

where

$$\mathbb{C}^3_p \; : \; x^{AB} \; = \; x_0^{AB} + \varepsilon^{ABCD}\mu_C\lambda_D$$

which is a totally null 3-plane.

Penrose Transform: H^3

- Then for $h \in \frac{1}{2}\mathbb{N}_0$

$$H^3(P, \mathcal{O}_P(-2h-4)) \;\cong\; \left\{ \begin{array}{c} \text{chiral zero-rest-mass fields} \\ \text{of spin } h \text{ on } M \end{array} \right\}$$

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- This can be interpreted as a contour integral

$$\psi_{A_1\cdots A_{2h}}(x) \;=\; \oint_\gamma \Omega^{(3,0)}\, \lambda_{A_1}\cdots\lambda_{A_{2h}} f_{-2h-4}(x\cdot\lambda, \lambda)\ ,$$

where γ is topologically a 3-torus and

$$\Omega^{(3,0)} \;:=\; \tfrac{1}{4!}\varepsilon^{ABCD}\lambda_A\,\mathrm{d}\lambda_B\wedge\mathrm{d}\lambda_C\wedge\mathrm{d}\lambda_D\ .$$

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What about $h < 0$?

Penrose–Ward Transform: H^2

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  by means of a Penrose–Ward transform.
- Note that in the case of interest for the self-dual 3-forms, we have $h = 1$ and thus $H^2(P, \mathcal{O}_P)$, which in turn is isomorphic to $H^2(P, \mathcal{O}_P^*)$. Hence, holomorphic bundle 1-gerbes on twistor space correspond to self-dual 3-form fields on space-time.

Twistor Action

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- Define holomorphic volume form on P

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where $\Omega^{(4,0)}(z) := \frac{1}{4!}\varepsilon_{ABCD}\,\mathrm{d}z^A\wedge\mathrm{d}z^B\wedge\mathrm{d}z^C\wedge\mathrm{d}z^D$ and $\Omega^{(3,0)}(\lambda) := \frac{1}{4!}\varepsilon^{ABCD}\lambda_A\,\mathrm{d}\lambda_B\wedge\mathrm{d}\lambda_C\wedge\mathrm{d}\lambda_D$.

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- Then,

$$S \;=\; \int \Omega^{(6,0)}\wedge B^{(0,2)}_{2h-2}\wedge\bar\partial C^{(0,3)}_{-2h-4}\;.$$

Non-Abelian Extensions And Supersymmetry

1205.3108 (CMP) with C Sämann

Principal Bundles—A Recap

- Let $M = \bigcup_a U_a$ be a manifold and G a Lie group. A principal G-bundle over M with connection is described by a G-valued Deligne 1-cocycle $(\{g_{ab}\}, \{A_a\})$ with

$$g_{ab}g_{bc} \;=\; g_{ac}\;, \quad A_b \;=\; g_{ab}^{-1}A_a g_{ab} + g_{ab}^{-1}\,\mathrm{d}g_{ab}\;.$$

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$$g_{ab}g_{bc} = g_{ac} \,, \quad A_b = g_{ab}^{-1} A_a g_{ab} + g_{ab}^{-1} \, \mathrm{d} g_{ab} \,.$$

- Two Deligne 1-cocycles $(\{g_{ab}\}, \{A_a\})$ and $(\{\tilde{g}_{ab}\}, \{\tilde{A}_a\})$ are said to be cohomologous whenever

$$g_a \tilde{g}_{ab} = g_{ab} g_b \,, \quad \tilde{A}_a = g_a^{-1} A_a g_a + g_a^{-1} \, \mathrm{d} g_a \,.$$

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- One associates a curvature 2-form $F_a := \mathrm{d}A_a + \frac{1}{2}[A_a, A_a]$ with

$$F_b = g_{ab}^{-1} F_a g_{ab} , \quad \tilde{F}_a = g_a^{-1} F_a g_a .$$

Question: How can one generalise this to incorporate gauge potentials of higher form-degree?

Lie Crossed Modules

- Let (G, H) a pair of Lie groups together with an automorphism action $\triangleright$ of G on H and a group homomorphism $\partial : H \to G$ such that

$$\partial(g \triangleright h) \;=\; g\partial(h)g^{-1}\;, \quad \partial(h_1) \triangleright h_2 \;=\; h_1 h_2 h_1^{-1}$$

called the equivariance and Peiffer conditions. This is known as a Lie crossed module.

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- A canonical example is the automorphism Lie 2-group $(G \xrightarrow{\partial} Aut(G), \triangleright)$ where ∂ is the embedding via conjugation and $\triangleright$ is the identity. For what follows, however, we need other examples.

Strict Principal 2-Bundles

Let $M = \bigcup_a U_a$ be a manifold. A strict principal 2-bundle with connective structure is described by a (G, H)-valued Deligne 2-cocycle $(\{g_{ab}\}, \{h_{abc}\}, \{A_a\}, \{B_a\}, \{\Lambda_{ab}\})$ with

$$
\begin{aligned}
t(h_{abc})g_{ab}g_{bc} &= g_{ac}\ , \\
h_{acd}h_{abc} &= h_{abd}(g_{ab} \triangleright h_{bcd})\ , \\
A_b &= g_{ab}^{-1}A_a g_{ab} + g_{ab}^{-1}\,\mathrm{d}g_{ab} - \partial(\Lambda_{ab})\ , \\
B_b &= g_{ab}^{-1} \triangleright B_a - \nabla_b\Lambda_{ab} - \tfrac{1}{2}\partial(\Lambda_{ab}) \triangleright \Lambda_{ab}\ , \\
\Lambda_{ac} &= \Lambda_{bc} + g_{bc}^{-1} \triangleright \Lambda_{ab} - g_{ac}^{-1} \triangleright (h_{abc}\nabla_a h_{abc}^{-1})\ .
\end{aligned}
$$

Strict Principal 2-Bundles

Two Deligne 2-cocycles $(\{g_{ab}\},\{h_{abc}\},\{A_a\},\{B_a\},\{\Lambda_{ab}\})$ and $(\{\tilde{g}_{ab}\},\{\tilde{h}_{abc}\},\{\tilde{A}_a\},\{\tilde{B}_a\},\{\tilde{\Lambda}_{ab}\})$ are said to be cohomologous whenever

$$
\begin{aligned}
g_a\tilde{g}_{ab} &= \partial(h_{ab})g_{ab}g_b\ ,\\
h_{ac}h_{abc} &= (g_a\triangleright\tilde{h}_{abc})h_{ab}(g_{ab}\triangleright h_{bc})\ ,\\
\tilde{A}_a &= g_a^{-1}A_ag_a+g_a^{-1}\,\mathrm{d}g_a-\partial(\Lambda_a)\ ,\\
\tilde{B}_a &= g_a^{-1}\triangleright B_a-\tilde{\nabla}_a\Lambda_a-\tfrac{1}{2}\partial(\Lambda_a)\triangleright\Lambda_a\ ,\\
\tilde{\Lambda}_{ab} &= g_b^{-1}\triangleright\Lambda_{ab}+\Lambda_b-\tilde{g}_{ab}^{-1}\triangleright\Lambda_a-(g_b^{-1}g_{ab}^{-1})\triangleright(h_{ab}^{-1}\nabla_b h_{ab})
\end{aligned}
$$

- The associated curvature 2- and 3-forms are

$$F_a := \mathrm{d}A_a + \tfrac{1}{2}[A_a, A_a] ,$$
$$H_a := \mathrm{d}B_a + A_a \triangleright B_a$$

with

$$F_b = g_{ab}^{-1} F_a g_{ab} - \partial(\nabla_b \Lambda_{ab} + \tfrac{1}{2}\partial(\Lambda_{ab}) \triangleright \Lambda_{ab}) ,$$
$$H_b = g_{ab}^{-1} \triangleright H_a - (F_b - \partial(B_b)) \triangleright \Lambda_{ab} ,$$

and

$$\tilde{F}_a = g_a^{-1} F_a g_a - \partial(\tilde{\nabla}_a \Lambda_a + \tfrac{1}{2}\partial(\Lambda_a) \triangleright \Lambda_a) ,$$
$$\tilde{H}_a = g_a^{-1} \triangleright H_a - (\tilde{F}_a - \partial(\tilde{B}_a)) \triangleright \Lambda_a .$$

- Thus, provided $F_a = \partial(B_a)$, the 3-form curvature transforms covariantly. This is called the fake curvature constraint.

Non-Abelian Self-Dual Tensor Field Equations

- Let us consider the following set of non-Abelian self-dual tensor equations

$$H = \mathrm{d}B + A \triangleright B\,, \quad H = \star_6 H\,, \quad F = \mathrm{d}A + \tfrac{1}{2}[A, A] = \partial(B)$$

on space-time $M \cong \mathbb{C}^6$. In spinor notation, this reads as

$$H^{AB} = \nabla^{C(A} B_C{}^{B)} = 0\,, \quad F_A{}^B = \partial(B_A{}^B)$$

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on space-time $M \cong \mathbb{C}^6$. In spinor notation, this reads as

$$H^{AB} = \nabla^{C(A} B_C{}^{B)} = 0 \,, \quad F_A{}^B = \partial(B_A{}^B)$$

- Can we use twistor theory to derive these equations including the just-mentioned gauge transformations from algebraic data on twistor space?

Penrose–Ward Transform: $P \stackrel{\pi_1}{\leftarrow} F \stackrel{\pi_2}{\rightarrow} M$

Theorem: There is a bijection between equivalence classes

(i) of holomorphic M-trivial strict principal 2-bundles on P,

(ii) of holomorphically trivial strict principal 2-bundles on F equipped with a flat relative connective structure, and

(iii) solutions to the non-Abelian self-dual tensor field equations on space-time M.

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Remark: The proof uses $H^1(F, \Omega^1_{\pi_1}) = 0$ and Riemann–Hilbert problems; the non-uniquess of RH problems is the origin of the gauge transformations on space-time.

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Question: What about supersymmetry?

Supertwistor Space

- Consider $\mathcal{N} = (n, 0)$ superspace $M = \mathbb{C}^{6|8n}$ with coordinates (x^{AB}, η_I^A) with $I, J, \ldots = 1, \ldots, 2n$. The derivatives

$$P_{AB} := \frac{\partial}{\partial x^{AB}} , \quad D_A^I := \frac{\partial}{\partial \eta_I^A} - 2\Omega^{IJ}\eta_J^B \frac{\partial}{\partial x^{AB}}$$

  obey

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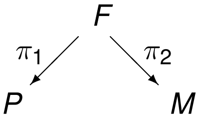
- Define the correspondence space F to be $F := \mathbb{C}^{4|8n} \times \mathbb{P}^3$ with coordinates $(x^{AB}, \eta_I^A, \lambda_A)$.
- Introduce a rank-$3|6n$ distribution $\langle V^A, V^{I\,AB}\rangle \hookrightarrow TF$ by $V^A := \lambda_B \partial^{AB}$ and $V^{I\,AB} = \frac{1}{2}\varepsilon^{ABCD}\lambda_C D_D^I$ which is integrable. Hence, we have foliation $P := F/\langle V^A, V^{I\,AB}\rangle$.

Supertwistor Space

- On P, we may use coordinates (z^A, η_I, λ_A) with $z^A\lambda_A = \Omega^{IJ}\eta_I\eta_J$ and thus



with π_2 being the trivial projection and

$$\pi_1 : (x^{AB}, \eta_I^A, \lambda_A) \;\mapsto\; (z^A, \eta_I, \lambda_A) \;=$$
$$= ((x^{AB} + \Omega^{IJ}\eta_I^A\eta_J^B)\lambda_B, \eta_I^A\lambda_A, \lambda_A)$$

- A point $x \in M$ corresponds to a complex projective 3-space in P, while a point $p \in P$ corresponds to a $3|6n$-superplane with

$$x^{AB} \;=\; x_0^{AB} + \varepsilon^{ABCD}\mu_C\lambda_D + 2\Omega^{IJ}\varepsilon^{CDE[A}\lambda_C\theta_{IDE}\eta_{0\,J}^{\;B]}\;,$$
$$\eta_I^A \;=\; \eta_{0\,I}^{\;A} + \varepsilon^{ABCD}\lambda_B\theta_{ICD}\;.$$

Theorem: There is a bijection between equivalence classes

(i) of holomorphic *M*-trivial strict principal 2-bundles on P and

(ii) of solutions to the constraint system

$$
\begin{aligned}
F_A{}^B &= \partial(B_A{}^B)\,, \quad F_{AB}{}^I_C = \partial(B_{AB}{}^I_C)\,, \quad F^{IJ}_{AB} = \partial(B^{IJ}_{AB})\,,\\
H^{AB} &= 0\,,\\
H_A{}^{BI}{}_C &= \delta^B_C \psi^I_A - \tfrac{1}{4}\delta^B_A \psi^I_C\,,\\
H_{AB}{}^{IJ}_{CD} &= \varepsilon_{ABCD}\phi^{IJ}\,,\\
H^{IJK}_{ABC} &= 0\,,
\end{aligned}
$$

on the chiral superspace M.

Remarks

- We obtain the fields $(H_{AB}, \psi^I_A, \phi^{IJ})$ which transform on-shell under gauge transformations as

$$(H_{AB}, \psi^I_A, \phi^{IJ}) \ \mapsto \ g^{-1} \triangleright (H_{AB}, \psi^I_A, \phi^{IJ}).$$

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- For $n = 1$ ($n = 2$), the multiplet $(H_{AB}, \psi^I_A, \phi^{IJ})$ constitutes an $\mathcal{N} = (n, 0)$ tensor multiplet consisting of 1 self-dual 3-form, 2 (4) Weyl spinors, and 1 (5) scalar(s). Note that for $n = 2$, the constraint $\Omega_{IJ}\phi^{IJ} = 0$ is automatically built in due to Bianchi identities (contrary to $\mathcal{N} = 4$ SYM in 4d)

Generalisations

1305.4870 (LMP) with C Sämann

1403.7188 (submitted) with B Jurčo and C Sämann

Strict Principal 3-Bundles

- The constraint $\partial(H)=0$ for the 3-form curvature H implies that it takes values in the centre of $\mathfrak{h}$.

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- A way to relax this is to categorify to the next level and work with strict principal 3-bundles which are modelled on Lie 2-crossed modules $L \overset{\partial}{\to} H \overset{\partial}{\to} G$.
- In turn, these bundles come with 1-, 2- and 3-form gauge potentials A, B, and C taking values in $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ with associated curvature forms

$$F := \mathrm{d}A + \tfrac{1}{2}[A,A]\,, \quad H := \mathrm{d}B + A \triangleright B\,,$$
$$G := \mathrm{d}C + A \triangleright C + \{B,B\}\,,$$

where $\{\cdot,\cdot\} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{l}$ is the Peiffer lifting.

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- Again, the Penrose–Ward transform boils down to changing the corresponding Deligne cocycles via boundary transformations which works due to the vanishing of $H^1(F, \Omega^2_{\pi_1})$ and $H^1(F, \Omega^2_{\pi_1})$.

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- Note that in certain cases, these theories accommodate some of the tensor hierarchy models of Samtleben, Sezgin & Wimmer.

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- Let $M = \bigcup_a U_a$ and define the Čech groupoid the groupoid with the set of objects $\dot{\bigcup}_a U_a$ and the set of morphisms $\dot{\bigcup}_{a,b} U_a \cap U_b$. Let $\mathrm{B}G$ be the groupoid which has only one object and the elements of G as morphisms. Then, principal G-bundles can be viewed as functors from the Čech groupoid to $\mathrm{B}G$.

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- We generalise this by defining weak principal k-bundles as weak k-functors from the Čech k-groupoid to $\mathsf{B}G$ for weak Lie k-groups G.

- Specifically, for $k = 2$: weak 2-category $\Rightarrow$ weak 2-groupoid $\Rightarrow$ weak 2-group $\Rightarrow$ semistrict 2-group $\Rightarrow$ semistrict Lie 2-group (only the associator remains non-trivial).

Semistrict Principal 2-Bundles

- Specifically, for $k = 2$: weak 2-category $\Rightarrow$ weak 2-groupoid $\Rightarrow$ weak 2-group $\Rightarrow$ semistrict 2-group $\Rightarrow$ semistrict Lie 2-group (only the associator remains non-trivial).
- We define semistrict principal 2-bundles as a weak 2-functors from the Čech 2-groupoid to the delooping $\mathrm{B}G$ of semistrict Lie 2-groups G.

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- We define semistrict principal 2-bundles as a weak 2-functors from the Čech 2-groupoid to the delooping $\mathsf{B}G$ of semistrict Lie 2-groups G.
- Differentiating G a la Ševera yields the corresponding semistrict Lie 2-algebra (2-term L_∞): one considers the functor from the category of smooth manifolds M to the category of G-valued descent data on surjective submersions $\mathbb{R}^{0|1} \times M \to M$

Semistrict Principal 2-Bundles

- Correspondingly, one finds 1-form A and 2-form B gauge potentials with the curvatures

$$F := \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) = \mu_1(B) ,$$
$$H := \mathrm{d}B + \mu_2(A, B) - \tfrac{1}{3!}\mu_3(A, A, A) ,$$

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- This construction also yields the full set of non-linear gauge transformation by means of equivalence transformations between functors.

- This allows us to formulate explicitly semistrict degree-2 Deligne cohomology: semistrict principal 2-bundles with connective structure are characterised by cocycles $(\{n_{abc}\},\{m_{ab}\},\{A_a\},\{B_a\},\{\Lambda_{ab}\})$ subject to equivalence; note that $F_a = s(B_a)$.

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$$H \;=\; \mathrm{d}B + \mu_2(A,B) - \tfrac{1}{3!}\mu_3(A,A,A) \;=\; \star_6 H\;.$$

plus supersymmetry.

Conclusions And Outlook

Summary

In general, we have seen that the area of twistor geometry and categorified principal bundles can be fruitfully combined to formulate self-dual higher gauge theory in six dimensions.

The advantage of twistor geometry is that the e.o.m. and the gauge transformations follow directly from complex algebraic data on twistor space.

Many open questions remain such as what higher gauge groups should be chosen, explicit solutions should be constructed, dimensional reductions should be performed, etc

Thank You!