Exact results in AdS/CFT from localization

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Field theory:

- In the last few years it has been appreciated that one can put general (Euclidean) supersymmetric gauge theories on curved backgrounds, preserving supersymmetry.
- In such a theory the VEV of any BPS operator *localizes*

$$
\begin{aligned}
\langle \mathcal{O}_{\mathsf{BPS}} \rangle \quad &= \quad \int_{\text{all fields}} \mathrm{e}^{-\mathsf{S}}\, \mathcal{O}_{\mathsf{BPS}} \\
&\overset{\text{exactly}}{=} \quad \int_{\mathcal{Q}-\text{invariant fields}} \mathrm{e}^{-\mathsf{S}}\, \mathcal{O}_{\mathsf{BPS}} \cdot (\text{one-loop determinant})\ .
\end{aligned}
$$

A form of *fixed point theorem*: $\mathcal{Q}$ is a supercharge, generating a supersymmetry variation of the theory.

- For appropriate classes of theories and operators one can compute such quantities *exactly* in field theory, on an *arbitrary* background $\mathbf{M_d}$.
- Applications include non-perturbative tests of various conjectured dualities.

In particular, if the field theory on (conformally) flat space has an AdS dual, we may try to compare these computations to gravity.

Gravity:

- There are large classes of supersymmetric gauge theories that, in a suitable large $\mathbf{N}$ limit, are conjectured to be described by the supergravity limit of string/M-theory.
- Typically described by a (warped) product $\mathrm{AdS}_{\mathbf{d+1}} \times \mathbf{Y}$, where different choices of internal space $\mathbf{Y}$ correspond to different gauge theories, and $\mathbf{N} =$ flux quantum number.
- We must then solve a supergravity filling problem in Euclidean quantum gravity: find the (least action) solution on some $\mathbf{M_{d+1}}$ such that $\partial \mathbf{M_{d+1}} = \mathbf{M_d}$.

I will summarize results for $\mathbf{d = 3}$ and $\mathbf{d = 5}$.

One can put an arbitrary $\mathcal{N} = 2$ supersymmetric gauge theory in $\mathbf{d} = \mathbf{3}$ dimensions on a (Euclidean) curved background following [Festuccia-Seiberg]: couple the theory to $\mathbf{d} = \mathbf{3}$ supergravity, and take a rigid limit in which $\mathbf{m_{Planck}} \to \infty$ [Closset-Dumitrescu-Festuccia-Komargodski].

As well as the background metric on $\mathbf{M_3}$, there are two background vector fields $\mathbf{A}$ and $\mathbf{V}$, and a scalar function $\mathbf{h}$, together with Killing spinor χ satisfying

$$(\nabla_\mu - \mathrm{i}\mathbf{A}_\mu)\chi \ = \ -\tfrac{\mathrm{i}}{2}\mathbf{h}\gamma_\mu\chi - \mathrm{i}\mathbf{V}_\mu\chi - \tfrac{1}{2}\epsilon_{\mu\nu\rho}\mathbf{V}^\nu\gamma^\rho\chi \ .$$

Of central importance for us is the Killing vector

$$\mathbf{K} \ = \ \chi^\dagger\gamma^\mu\chi\partial_\mu \ = \ \partial_\psi \ .$$

The vector field $\mathbf{K}$ is nowhere zero, generating a foliation of $\mathbf{M_3}$ which is transversely holomorphic. The metric is locally

$$\mathrm{d}s_3^2 \ = \ \Omega(z,\bar{z})^2(\mathrm{d}\psi + a)^2 + c(z,\bar{z})^2 \mathrm{d}z\mathrm{d}\bar{z} \ .$$

where $\mathbf{z}$ is a complex coordinate.

Essentially the background is parametrized by an arbitrary choice of the functions $\Omega(z,\bar{z}), c(z,\bar{z})$, and local one-form $a = a(z,\bar{z})\mathrm{d}z + \mathrm{c.c.}$, and imposing the Killing spinor equation then fixes everything else in terms of these.

If all the orbits of K close then $\mathbf{M_3}$ is the total space of a $\mathbf{U(1)}$ orbibundle over an orbifold Riemann surface Σ (a Seifert fibred 3-manifold).

On the other hand, if at least one orbit is open then $\mathbf{M_3}$ necessarily admits a $\mathbf{U(1)} \times \mathbf{U(1)}$ isometry, and we may write

$$\mathsf{K} \;=\; \partial_\psi \;=\; \mathbf{b_1}\partial_{\varphi_1} + \mathbf{b_2}\partial_{\varphi_2} \;,$$

where $\mathbf{b_1}, \mathbf{b_2} \neq \mathbf{0}$ can be thought of as parametrizing a choice of K.

General $\mathcal{N} = 2$ supersymmetric gauge theory in $\mathbf{d = 3}$ dimensions:

- Vector multiplet $(\mathscr{A}, \sigma, \lambda, \mathrm{D})$ in the adjoint of the gauge group $\mathbf{G}$, for which we may write a Chern-Simons, as well as Yang-Mills, action.
- Matter chiral multiplet (ϕ, ψ, F) in a representation $\mathcal{R}$ of $\mathbf{G}$, with superpotential.

The localization computation for this general set-up is in our paper [1307.6848]. One first determines the $\mathcal{Q}$-invariant field configurations, and then computes the one-loop determinants around these.

In the vector multiplet we find the localization equations for $\mathbf{M_3} \cong \mathbf{S^3}$ imply

$$\mathscr{A} \;=\; 0\,, \qquad \Omega\sigma \;=\; \sigma_0 \;=\; \text{constant}\,, \qquad \mathsf{D} = -\frac{\mathsf{h}}{\Omega}\sigma_0\,.$$

The matter multiplet is trivial: all fields localize to zero.

The classical action for $\mathbf{M_3} \cong \mathbf{S^3}$, evaluated on the localization locus, is given entirely by the Chern-Simons action:

$$\mathsf{S}_{\mathrm{CS}} \;=\; -\frac{\mathrm{i}\mathsf{k}}{2\pi}\mathrm{Tr}(\sigma_0^2)\int_{\mathsf{M}_3}\frac{\mathsf{h}}{\Omega^2}\sqrt{\det \mathsf{g}}\,\mathrm{d}^3\mathsf{x} \;=\; \frac{\mathrm{i}\pi\mathsf{k}}{|\mathsf{b}_1\mathsf{b}_2|}\mathrm{Tr}(\sigma_0^2)\,.$$

Most of the work is in computing the one-loop determinants.

The final result for the partition function is

$$\mathbf{Z} = \int d\sigma_0 \, e^{-\frac{i\pi k}{|b_1 b_2|} \mathrm{Tr}\, \sigma_0^2} \prod_{\alpha \in \Delta_+} 4 \sinh \frac{\pi \sigma_0 \alpha}{|b_1|} \sinh \frac{\pi \sigma_0 \alpha}{|b_2|} \cdot \prod_{\rho} s_\beta \left[\frac{i(\beta + \beta^{-1})}{2}(1 - R) - \frac{\rho(\sigma_0)}{\sqrt{|b_1 b_2|}} \right] .$$

Here we have defined $\beta = \sqrt{|b_1 / b_2|}$, ρ denote weights in a weight space decomposition of the representation $\mathcal{R}$ for the matter fields, **R** is their R-charge, and $s_\beta(z)$ denotes the double sine function.

It is also straightforward to insert BPS operators, for example the Wilson loop

$$\mathbf{W} \;=\; \mathrm{Tr}_{\mathcal{R}}\left[\mathcal{P}\exp\int_\gamma \mathrm{ds}(\mathrm{i}\mathscr{A}_\mu \dot{\mathrm{x}}^\mu + \sigma|\dot{\mathrm{x}}|)\right] \ ,$$

where $\mathbf{x}^\mu(\mathbf{s})$ parametrizes with worldline $\gamma =$ orbit of $\mathbf{K}$, is $\mathcal{Q}$-invariant.

$\langle\, \mathbf{W} \,\rangle$ is then computed by inserting $\mathrm{Tr}_{\mathcal{R}}\mathrm{e}^{2\pi\ell\sigma_0}$ into the localized partition function, where $2\pi\ell =$ length of Reeb orbit (*e.g.* at the "pole" where $\partial_{\varphi_1} = 0$, $\ell = 1/|\mathbf{b}_2|$) [Farquet-JFS].

For comparison with AdS/CFT we should focus on field theories that in (conformally) flat space have an AdS gravity dual.

There are huge classes of these, described by Chern-Simons-quiver gauge theories, with $\mathbf{U(N)^p}$ gauge groups, *e.g.* the maximally supersymmetric case is the ABJM theory, living on $\mathbf{N}$ M2-branes in flat space.

The gravity duals are M-theory backgrounds of the form $\mathrm{AdS}_4 \times \mathbf{Y_7}$, with $\mathbf{N}$ units of $*\mathbf{G_4}$ through the internal space $\mathbf{Y_7}$, and arise as *e.g.* near-horizon limits of $\mathbf{N}$ M2-branes at Calabi-Yau four-fold singularities [Martelli-JFS, many other authors].

The large $\mathbf{N}$ limit of the matrix model partition function was computed in [Martelli-Passias-JFS], using a saddle point method of [Herzog-Klebanov-Pufu-Tesileanu].

This involves the asymptotic expansion of the double sine function, and an ansatz for the saddle point eigenvalue distribution for $\boldsymbol{\sigma_0}$.

The final results are extremely simple:

$$\begin{aligned} \log Z &= \frac{(|b_1| + |b_2|)^2}{4|b_1 b_2|} \cdot \log Z_{\text{round } S^3} , \\ \log\langle W \rangle &= \tfrac{1}{2}\ell(|b_1| + |b_2|) \cdot \log\langle W \rangle_{\text{round } S^3} . \end{aligned}$$

In particular, the dependence on the background geometry factorizes from the dependence on the choice of gauge theory.

In [1404.0268] and [1406.2493] we have reproduced these formulas from a dual Euclidean quantum gravity calculation, for a very general class of solutions.

We work in $\mathcal{N} = 2$ gauged supergravity in four dimensions. This is Einstein-Maxwell theory, with a graviphoton $\mathcal{A}$, and we use the fact that any supersymmetric solution of this theory on $\mathbf{M_4}$ uplifts to a supersymmetric solution of M-theory on $\mathbf{M_4} \times \mathbf{Y_7}$ [Gauntlett-Varela].

The Killing spinor equation takes the form

$$\left[\nabla_\mu - \mathrm{i}\mathcal{A}_\mu + \tfrac{1}{2}\Gamma_\mu + \tfrac{\mathrm{i}}{4}\mathcal{F}_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right]\epsilon \ = \ 0 \ .$$

The local form of Euclidean supersymmetric solutions to this theory was studied by [Dunajski-Gutowski-Sabra-Tod].

In particular, there is a class of *self-dual* solutions in which $*_4\mathcal{F} = -\mathcal{F}$ is anti-self-dual, and the four-metric is then Einstein with anti-self-dual Weyl tensor.

We also have a Killing vector

$$\mathsf{K} \ = \ \mathrm{i}\epsilon^\dagger \Gamma^\mu \Gamma_5 \epsilon \partial_\mu \ = \ \partial_\psi \ .$$

Self-dual Einstein metrics with a Killing vector have a rich geometric structure. They are (locally) conformal to a scalar-flat Kähler metric, with the metric determined entirely by a solution to the Toda equation:

$$\mathrm{d}s_4^2 \;=\; \frac{1}{y^2}\mathrm{d}s^2_{\mathrm{Kahler}} \;=\; \frac{1}{y^2}\Big[V^{-1}(\mathrm{d}\psi+\phi)^2 + V(\mathrm{d}y^2 + 4e^{w}\mathrm{d}z\mathrm{d}\bar{z})\Big]\;.$$

where $V = 1 - \frac{1}{2}y\partial_y w$, the expression for $\mathrm{d}\phi$ is known (but complicated), and

$$\partial_z\partial_{\bar{z}}w + \partial_y^2 e^{w} \;=\; 0\;.$$

The conformal boundary is at $\mathbf{y = 0}$, and one can show that the structure induced on the conformal boundary is precisely the three-dimensional background geometry of [Closset-Dumitrescu-Festuccia-Komargodski].

In particular

$$\epsilon \;=\; \mathbf{y}^{-1/2}\left[\left(\mathbf{1} + \Gamma_0 + \tfrac{1}{4}\mathbf{y}\mathbf{w}_{(1)}\Gamma_0\right)\begin{pmatrix} \chi \\ \mathbf{0} \end{pmatrix} + \mathcal{O}(\mathbf{y}^2)\right] \ ,$$

where χ is a three-dimensional spinor satisfying the Killing spinor equation we saw earlier, and we expand $\mathbf{w}(\mathbf{y}, \mathbf{z}, \bar{\mathbf{z}}) = \mathbf{w}_{(0)}(\mathbf{z}, \bar{\mathbf{z}}) + \mathbf{y}\mathbf{w}_{(1)}(\mathbf{z}, \bar{\mathbf{z}}) + \mathcal{O}(\mathbf{y}^2)$.

Suppose we have such a solution. The holographic free energy is

$$-\log Z_{\text{gravity}} \;=\; S_{\text{Einstein-Maxwell}} + S_{\text{Gibbons-Hawking}} + S_{\text{counterterms}} \;.$$

The individual terms certainly depend on the detailed solution. For example

$$\frac{1}{16\pi G_N}\int_{B_4} F^2\sqrt{\det g}\,\mathrm{d}^4x \;=\; -\frac{\pi(|b_1+b_2|)^2}{8G_N|b_1b_2|} + \frac{1}{256\pi G_N}\int_{M_3}\left(3w_{(1)}^3 + 4w_{(1)}w_{(2)}\right)\sqrt{\det g_3}\,\mathrm{d}^3x \;.$$

Here we have assumed the topology $M_3 \cong S^3$ and $M_4 \cong B_4$.

However, the final result is

$$-\log Z_{\text{gravity}} = \frac{(|b_1| + |b_2|)^2}{4|b_1 b_2|} \cdot \frac{\pi}{2G_N} ,$$

agreeing with the field theory computation!

The Wilson loop in the fundamental representation maps to a supersymmetric M2-brane, wrapping a calibrated copy of the M-theory circle [Farquet-JFS], and with a minimal surface $\Sigma \subset B_4$ with $\partial\Sigma = \gamma =$ orbit of Reeb vector K.

$\log\langle W \rangle_{\text{gravity}}$ is identified with minus the regularized action of the M2-brane, and in [1406.2493] we showed this reproduces the large N field theory result.

We now change focus to $\mathbf{d = 5}$. [Imamura] has defined five-dimensional supersymmetric gauge theories on the $\mathbf{SU(3) \times U(1)}$-invariant squashed five-sphere background

$$\mathbf{ds_5^2 \; = \; \frac{1}{s^2}(d\tau + C)^2 + ds^2_{\mathbb{CP}^2}}$$

where $\frac{1}{2}\mathbf{dC} = \omega =$ Kähler form for the Fubini-Study metric on $\mathbb{CP}^2$. Here $\mathbf{s} =$ squashing parameter, with $\mathbf{s = 1}$ the round five-sphere.

There is also a background R-symmetry gauge field

$$\mathbf{A^R \; = \; \frac{1}{s^2}(1 + Q\sqrt{1-s^2})\sqrt{1-s^2}(d\tau + C)} \; ,$$

where $\mathbf{U(1)_R \subset SU(2)_R}$ and $\mathbf{Q = 1}$, $\mathbf{Q = -3}$ give rise to 3/4 BPS and 1/4 BPS solutions, respectively.

The *perturbative* partition function again localizes onto an integral over the constant mode σ_0 of the scalar in the vector multiplet, and the final formula involves triple sine functions.

A particular class of five-dimensional gauge theories, with gauge group $\mathbf{USp(2N)}$ and arising from a D4-D8 system, is expected to have a large $\mathbf{N}$ description in terms of massive type IIA supergravity [Ferrara-Kehagias-Partouche-Zaffaroni], [Brandhuber-Oz].

In [Jafferis-Pufu] the large $\mathbf{N}$ limit of the partition function of these theories on the *round* sphere was computed and successfully compared to the entanglement entropy of the dual warped $\mathrm{AdS}_6 \times \mathbf{S}^4$ supergravity solution.

In [1405.7194] we computed the large $\mathbf{N}$ limit of the $\mathbf{USp(2N)}$ gauge theories on the squashed five-sphere, finding the free energy

$$\log Z = \frac{(|b_1| + |b_2| + |b_3|)^3}{27|b_1 b_2 b_3|} \cdot \log Z_{\text{round } S^5} ,$$

where

$$\begin{cases} b_1 = b_2 = b_3 & 1/4 \text{ BPS} \\ b_1 = -1 - \sqrt{1 - s^2} \, , \; b_2 = b_3 = 1 - \sqrt{1 - s^2} & 3/4 \text{ BPS} \end{cases}$$

There is again a supersymmetric Killing vector bilinear $\mathbf{K}$, and embedding $S^5 \subset \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$, this is $K = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} + b_3 \partial_{\varphi_3}$.

We also computed the large $\mathbf{N}$ limit of BPS Wilson loops. If the worldline wraps the $\mathbf{S}^1_i \subset \mathbf{S}^5$ at the origin of two copies of $\mathbb{R}^2$, then we find

$$\log\langle \mathbf{W} \rangle = \frac{|\mathbf{b}_1| + |\mathbf{b}_2| + |\mathbf{b}_3|}{3|\mathbf{b}_i|} \cdot \log\langle \mathbf{W} \rangle_{\text{round } \mathbf{S}^5} .$$

We have reproduced these formulae from a dual supergravity computation.

We work in six-dimensional Romans $\mathbf{F(4)}$ gauged supergravity, which is a consistent truncation of massive IIA supergravity on $\mathbf{S}^4$ [Cvetic-Lu-Pope]. As well as the metric, there is a scalar $\mathbf{X}$, two-form potential $\mathbf{B}$, one-form potential $\mathbf{A}$, and an $\mathbf{SO(3)} \sim \mathbf{SU(2)}$ R-symmetry gauge field $\mathbf{A}_I$, $\mathbf{I} = 1, 2, 3$.

The one-form $\mathbf{A}$ is a Stueckelberg field, which may be set to $\mathbf{A}=\mathbf{0}$ by a gauge transformation. The $\mathbf{B}$-field then becomes massive, and the Euclidean action is

$$\begin{aligned} S_{\rm bulk} &= -\frac{1}{16\pi G_N}\int_{M_6}\Big[R*1-4X^{-2}\mathrm{d}X\wedge*\mathrm{d}X \\ &\quad -\left(\tfrac{2}{9}X^{-6}-\tfrac{8}{3}X^{-2}-2X^{2}\right)*1-\tfrac{1}{2}X^{-2}\left(\tfrac{4}{9}B\wedge*B+F_I\wedge*F_I\right) \\ &\quad -\tfrac{1}{2}X^{4}H\wedge*H-\mathrm{i}B\wedge\left(\tfrac{2}{27}B\wedge B+\tfrac{1}{2}F_I\wedge F_I\right)\Big]\ . \end{aligned}$$

Notice the cubic Chern-Simons coupling for $\mathbf{B}$. Its curvature is $\mathbf{H}=\mathbf{dB}$.

A solution to the corresponding equations of motion is supersymmetric provided the Killing spinor equation and dilatino equation hold.

The squashed five-sphere background has $SU(3) \times U(1)$ symmetry, and one expects this to be preserved by the bulk filling. This leads to the ansatz

$$\begin{aligned} ds_6^2 &= \alpha^2(r)dr^2 + \gamma^2(r)(d\tau + C)^2 + \beta^2(r)ds^2_{\mathbb{CP}^2} , \\ B &= p(r)dr \wedge (d\tau + C) + \tfrac{1}{2}q(r)dC , \\ A_I &= f_I(r)(d\tau + C) , \end{aligned}$$

together with $X = X(r)$.

We have constructed smooth, supersymmetric, asymptotically locally Euclidean AdS solutions with the topology $M_6 \cong B_6$, with conformal boundary the squashed five-sphere backgrounds of [Imamura]. These may be given as expansions around the conformal boundary $r = \infty$, and/or as expansions in the squashing parameter s.

Reparametrization invariance allows us to set $\beta(r) = 3\sqrt{6r^2-1}/\sqrt{2}$ to its AdS_6 value, and an $SO(3)$ rotation sets $f_3(r) = f(r)$, $f_1(r) = f_2(r) = 0$.

For example, for the 3/4 BPS solution the first few terms in the expansion around $r = \infty$ are

$$\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}r + \frac{8+s^2}{36\sqrt{2}s^2}\frac{1}{r^3} + \ldots, \\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{-16+7s^2}{12\sqrt{3}s^3}\frac{1}{r} - \frac{-1280+1120s^2+241s^4}{2592\sqrt{3}s^5}\frac{1}{r^3} + \ldots, \\
X(r) &= 1 + \frac{1-s^2-3\sqrt{1-s^2}}{54s^2}\frac{1}{r^2} + \frac{s^2\sqrt{1-s^2}\kappa}{12\left(1-s^2+\sqrt{1-s^2}\right)}\frac{1}{r^3} + \ldots, \\
p(r) &= -\frac{i\sqrt{\frac{2}{3}}\left(s^2+3\sqrt{1-s^2}-1\right)}{s^3}\frac{1}{r^2} + \ldots, \\
q(r) &= -\frac{3i\left(\sqrt{6}\sqrt{1-s^2}\right)}{s}r + \frac{\sqrt{\frac{2}{3}}i\sqrt{1-s^2}\left(5s^2+9\sqrt{1-s^2}-5\right)}{3s^3}\frac{1}{r} + \ldots, \\
f(r) &= \frac{1-s^2+\sqrt{1-s^2}}{s^2} + \frac{2\left(-2+2s^2-(2+s^2)\sqrt{1-s^2}\right)}{9s^4}\frac{1}{r^2} + \frac{\kappa}{r^3} + \ldots .
\end{aligned}$$

The parameter κ is uniquely determined by requiring this to extend to a smooth solution on the ball $\mathbf{M_6} \cong \mathbf{B_6}$. As an expansion in

$$\delta = \sqrt{-1 + s^{-1}}$$

this is

$$\frac{3\sqrt{3}}{4}\kappa = \delta + \frac{\sqrt{2}}{3}\delta^2 + \frac{113}{36}\delta^3 + \frac{25}{9\sqrt{2}}\delta^4 + \frac{1127}{288}\delta^5 + \frac{35}{9\sqrt{2}}\delta^6 + \ldots$$

Similar results hold in the 1/4 BPS case, except here we find a *two-parameter* family of solutions, leading to a new supersymmetric squashing of $\mathbf{S^5}$. In particular this includes a one-parameter subfamily of 1/2 BPS solutions.

As in four dimensions the regularized action is

$$-\log \mathbf{Z}_{\mathbf{gravity}} \;=\; \mathbf{S}_{\mathbf{bulk}} + \mathbf{S}_{\mathbf{Gibbons-Hawking}} + \mathbf{S}_{\mathbf{ct}} \;.$$

However, unlike in four dimensions the counterterms $\mathbf{S}_{\mathbf{ct}}$ had not been computed.

This is a straightforward, but very long, computation. In particular the **B**-field is both massive and has a cubic Chern-Simons interaction, which leads to a much more complicated analysis than for more standard fields.

$$
\begin{aligned}
S_{ct} \quad = \quad & \frac{1}{8\pi G_N}\int_{\partial M_6}\Big\{\Big[\frac{4\sqrt{2}}{3}+\frac{1}{2\sqrt{2}}R(h)-\frac{1}{6\sqrt{2}}\|B\|_h^2+\frac{3}{4\sqrt{2}}R(h)_{ij}R(h)^{ij}-\frac{15}{64\sqrt{2}}R(h)^2-\frac{3}{4\sqrt{2}}\|F_I\|_h^2 \\
& +\frac{1}{12\sqrt{2}}\mathrm{Tr}_h B^4+\frac{5}{8\sqrt{2}}\|d *_h B+\frac{i\sqrt{2}}{3}B\wedge B\|_h^2-\frac{1}{4\sqrt{2}}\langle B, d\delta_h B+\frac{i\sqrt{2}}{3}d *_h B\wedge B\rangle_h-\frac{1}{\sqrt{2}}\|dB\|_h^2 \\
& +\frac{4\sqrt{2}}{3}(1-X)^2-\frac{1}{\sqrt{2}}\langle \mathrm{Ric}(h)\circ B, B\rangle_h+\frac{9}{32\sqrt{2}}R(h)\|B\|_h^2-\frac{13}{192\sqrt{2}}\|B\|_h^4\Big]\sqrt{\det h}\, d^5x \\
& -\frac{1}{4\sqrt{2}}B\wedge\Big[d *_h dB+\frac{\sqrt{2}i}{3}B\wedge\delta_h B-\frac{2}{9}B\wedge *_h(B\wedge B)\Big]\Big\} .
\end{aligned}
$$

Here $\mathrm{Ric}(h)_{ij} = R(h)_{ij}$ denotes the Ricci tensor of the boundary metric h_{ij}, with $R(h)$ the Ricci scalar. The inner product of two p-forms ν_1, ν_2 is defined by $\langle \nu_1, \nu_2\rangle_h \sqrt{\det h}\, d^5x = \nu_1 \wedge *_h \nu_2$, which then also defines the square norm via $\|\nu\|_h^2 = \langle \nu, \nu\rangle_h$. The adjoint δ_h of d with respect to h_{ij} acting on the two-form B is $\delta_h B = *_h d *_h B$, and we have also defined $\mathrm{Tr}_h B^4 \equiv B_i{}^j B_j{}^k B_k{}^l B_l{}^i$. Finally, we have defined the p-form $(S\circ\nu)_{i_1\cdots i_p} \equiv S_{[i_1}{}^j \nu_{|j|i_2\cdots i_p]}$, where S_{ij} is any symmetric 2-tensor, and ν is any p-form.

Using this we may compute the holographic free energy. For example, for the 3/4 BPS solution we find

$$\mathbf{S_{bulk}} + \mathbf{S_{Gibbons-Hawking}} + \mathbf{S_{ct}} = -\frac{27\pi^2}{4G_N}\left(1 + \frac{8}{3}\delta^2 + \frac{16\sqrt{2}}{27}\delta^3 + \frac{68}{27}\delta^4 + \frac{28\sqrt{2}}{27}\delta^5 + \frac{32}{27}\delta^6 + \ldots\right) .$$

This agrees with the field theory result. The BPS Wilson loop maps to a fundamental string in type **IIA**, at the "pole" of the internal $\mathbf{S^4}$ [Assel-Estes-Yamazaki]. The renormalized string action is

$$\mathbf{S_{string}} = \int_{\Sigma}\left[\mathbf{X}^{-2}\sqrt{\det\gamma}\,\mathbf{d^2x} + \mathbf{iB}\right] - \frac{3}{\sqrt{2}}\mathrm{length}(\partial\Sigma) ,$$

and also agrees with the large **N** field theory results.