Higher Gauge Theory and M-theory

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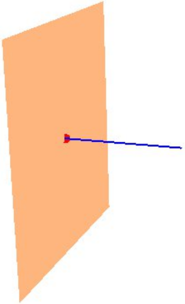
Based on work w. S Palmer, G Demessie, B Jurčo, M Wolf, P Ritter, R Szabo:

- Higher Gauge Theory: 1203.5757, 1308.2622, 1311.1977, 1406.5342
- Integrability: 1105.3904, 1205.3108, 1305.4870, 1312.5644, 1403.7185
- Geometric quantization: 1211.0395, 1308.4892

Why Higher Gauge Theory?

(2,0) theory should capture parallel transport of self-dual strings.

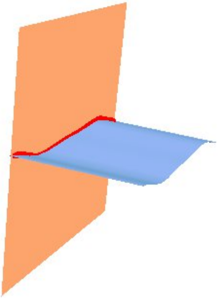


D-branes

- D-branes interact via strings.
- Effective description: theory of endpoints
- Parallel transport of these: Gauge theory

M5-branes

- M5-branes interact via M2-branes.
- Eff. description: theory of self-dual strings
- Parallel transport: Higher gauge theory
- (2,0) theory a higher gauge theory (HGT)?



So why not write down an HGT action and be done?

Things are more complicated...

- Higher gauge theory is a very young area (since ∼2002).
- Very few actions known for higher gauge theory.
- More groundwork needed (2-vector spaces, ...)

However, what we can see so far is very encouraging:

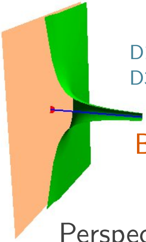
- Integrability of BPS subsectors via ADHM-type constructions
- Twistor descriptions of HGTs
- M2-brane models (BLG/ABJM) are HGTs
- (1,0)-models from tensor hierarchies are HGTs
- Noncommutativity lifts to nonassociativity
- IKKT model has a clear categorified analogue
- ...

Let's start at a pedestrian pace:

Lifting a D-brane configuration to M-theory

Monopoles and Self-Dual Strings

Lifting monopoles to M-theory yields self-dual strings.



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration

Perspective of D1:

Nahm eqn.

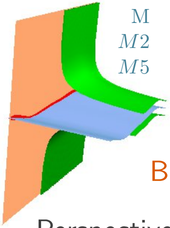
$$\frac{\mathrm{d}}{\mathrm{d}x^6}X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

↕ Nahm transform ↕

Perspective of D3:

Bogomolny monopole eqn.

$$F_{ij} = [\nabla_i, \nabla_j] = \varepsilon_{ijk}\nabla_k\Phi$$



M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

BPS configuration

Perspective of M2:

"Basu-Harvey eqn."

$$\frac{\mathrm{d}}{\mathrm{d}x^6}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

↕ generalized Nahm transform ↕

Perspective of M5:

Self-dual string eqn.

$$H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]} = \varepsilon_{\mu\nu\rho\sigma}\partial_\sigma\Phi$$

3-Lie algebra (do not confuse with Lie 3-algebras)

$\mathcal{A}$ is a vector space, $[\cdot,\cdot,\cdot]$ trilinear+antisymmetric.

Satisfies a "3-Jacobi identity," the fundamental identity:

$$[A,B,[C,D,E]] = [[A,B,C],D,E] + [C,[A,B,D],E] + [C,D,[A,B,E]]$$

Filippov (1985)

Algebra of inner derivations closes due to fundamental identity

$$D : \mathcal{A} \wedge \mathcal{A} \to \mathsf{Der}(\mathcal{A}) =: \mathfrak{g}_{\mathcal{A}} \qquad D(A,B) \rhd C := [A,B,C]$$

- 3-algebras $\overset{1:1}{\longleftrightarrow}$ metric Lie algebras $\mathfrak{g} \cong \mathsf{Der}(\mathcal{A})$ faithful orthog. representations $V \cong \mathcal{A}$
  J Figueroa-O'Farrill et al., 0809.1086
- They form strict Lie 2-algebras. S Palmer&CS, 1203.5757
- Hint: M2-brane models are linked to higher gauge theories.

Generalizing the ADHMN construction to M-branes

That is, find solutions to $H = \star \mathrm{d}\Phi$
from solutions to the Basu-Harvey equation.

An M5-brane seems to require ...

not an Albanian Gerbil, but an Abelian Gerbe

Principal U(1)-bundle over manifold M with cover $(U_i)_i$:

$$F \in \Omega^2(M, \mathfrak{u}(1)) \text{ with } \mathrm{d}F = 0$$
$$A_{(i)} \in \Omega^1(U_i, \mathfrak{u}(1)) \text{ with } F = \mathrm{d}A_{(i)}$$
$$g_{ij} \in \Omega^0(U_i \cap U_j, \mathsf{U}(1)) \text{ with } A_{(i)} - A_{(j)} = \mathrm{d}\log g_{ij}$$

E.g.: Dirac monopoles, principal U(1)-bundles over S^2, $c_1 \sim \int_{S^2} F$

Abelian (local) gerbe over manifold M with cover $(U_i)_i$:

$$H \in \Omega^3(M, \mathfrak{u}(1)) \text{ with } \mathrm{d}H = 0$$
$$B_{(i)} \in \Omega^2(U_i, \mathfrak{u}(1)) \text{ with } H = \mathrm{d}B_{(i)}$$
$$A_{(ij)} \in \Omega^1(U_i \cap U_j, \mathfrak{u}(1)) \text{ with } B_{(i)} - B_{(j)} = \mathrm{d}A_{ij}$$
$$h_{ijk} \in \Omega^0(U_i \cap U_j \cap U_k, \mathfrak{u}(1)) \text{ with } A_{(ij)} - A_{(ik)} + A_{(jk)} = \mathrm{d}h_{ijk}$$

E.g.: Self-dual strings, abelian gerbes over S^3, $d_1 \sim \int_{S^3} H$

Gerbes are somewhat unfamiliar, difficult to work with.
(at least for physicists)

Can we somehow avoid using gerbes?

Abelian Gerbes and Loop Space

By going to loop space, one can reduce differential forms by one degree.

Consider the following double fibration:

$$\begin{array}{ccccc} & & \mathcal{L}M \times S^1 & & \\ & \stackrel{ev}{\swarrow} & & \stackrel{pr}{\searrow} & \\ M & & & & \mathcal{L}M \end{array}$$

Identify $T\mathcal{L}M = \mathcal{L}TM$, then: $x \in \mathcal{L}M \Rightarrow \dot{x}(\tau) \in T\mathcal{L}M$

Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \to \Omega^k(\mathcal{L}M)~, \quad v_i = \oint \mathrm{d}\tau\, v_i^\mu(\tau) \frac{\delta}{\delta x^\mu(\tau)} \in T\mathcal{L}M$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \oint_{S^1} \mathrm{d}\tau\, \omega(x(\tau))(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau))$$

Nice properties: reparameterization invariant, chain map, ...

An abelian local gerbe over M is a principal $\mathsf{U}(1)$-bundle over $\mathcal{L}M$.

Transgressed Self-Dual Strings

By going to loop space, one can reduce differential forms by one degree.

Recall the self-dual string equation on $\mathbb{R}^4$: $H_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\lambda}\frac{\partial}{\partial x^\lambda}\Phi$

Its transgressed form is an equation for a 2-form F on $\mathcal{L}\mathbb{R}^4$:

$$F_{(\mu\sigma)(\nu\rho)} = \delta(\sigma-\rho)\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\tau)\,\frac{\partial}{\partial y^\lambda}\Phi(y)\Big|_{y=x(\tau)}$$

Extend to full non-abelian loop space curvature:

$$\begin{aligned} F^{\pm}_{(\mu\sigma)(\nu\tau)} =& \big(\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi\big)_{(\sigma\tau)} \\ &\mp \big(\dot{x}_\mu(\sigma)\nabla_{(\nu\tau)}\Phi + \dot{x}_\nu(\sigma)\nabla_{(\mu\tau)}\Phi - \delta_{\mu\nu}\dot{x}^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\big)_{[\sigma\tau]} \end{aligned}$$

where $\nabla_{(\mu\sigma)} := \oint \mathrm{d}\tau\,\delta x^\mu(\tau)\wedge\left(\frac{\delta}{\delta x^\mu(\tau)} + A_{(\mu\tau)}\right)$

Goal: Construct solutions to this equation.

The ADHMN Construction

The ADHMN construction nicely translates to self-dual strings on loop space.

Nahm transform: Instantons on $T^4 \mapsto$ instantons on $(T^4)^*$

Roughly here:

$$T^4\colon \begin{cases} 3 \text{ rad. } 0 \\ 1 \text{ rad. } \infty \ : \ \text{D1 WV} \end{cases} \quad \text{and} \quad (T^4)^*\colon \begin{cases} 3 \text{ rad. } \infty \ : \ \text{D3 WV} \\ 1 \text{ rad. } 0 \end{cases}$$

Dirac operators: X^i solve Nahm eqn., X^μ solve Basu-Harvey eqn.

$$\text{IIB}: \quad \nabla\!\!\!/ = -\mathbb{1}\frac{\mathrm{d}}{\mathrm{d}x^6} + \sigma^i(\mathrm{i}X^i + x^i \mathbb{1}_k)$$

$$\text{M}: \quad \nabla\!\!\!/ = -\gamma_5 \frac{\mathrm{d}}{\mathrm{d}x^6} + \tfrac{1}{2}\gamma^{\mu\nu}\left(D(X^\mu, X^\nu) - \mathrm{i}\oint \mathrm{d}\tau\, x^\mu(\tau)\dot{x}^\nu(\tau)\right)$$

normalized zero modes: $\bar{\nabla\!\!\!/}\psi = 0$ and $\mathbb{1} = \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}\psi$

Solution to Bogomolny/self-dual string equations:

$$A := \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}\,\mathrm{d}\,\psi \quad \text{and} \quad \Phi := -\mathrm{i}\int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}\, s\, \psi$$

Remarks on the Construction

The construction is very natural and behaves as expected.

- Can easily make the discussion non-abelian.
- Nahm eqn. and Basu-Harvey eqn. play analogous roles.
- Construction extends to general. Basu-Harvey eqn. (ABJM).
- One can construct many examples explicitly.
- It reduces perfectly to ADHMN via the M2-Higgs mechanism.

CS, 1007.3301, S Palmer&CS, 1105.3904

However:

Loop spaces are scary...

So, let’s bite the bullet:

Nonabelian Gerbes and Higher Gauge Theory

Parallel transport of particles in representation of gauge group G:

- holonomy functor hol : path $p \mapsto \mathsf{hol}(p) \in \mathsf{G}$
- $\mathsf{hol}(p) = P\exp(\int_p A)$, P: path ordering, trivial for $\mathsf{U}(1)$.

Parallel transport of strings with gauge group $\mathsf{U}(1)$:

- map hol : surface $s \mapsto \mathsf{hol}(s) \in \mathsf{U}(1)$
- $\mathsf{hol}(s) = \exp(\int_s B)$, B: connective structure on gerbe.

Nonabelian case:

- much more involved!
- no straightforward definition of surface ordering
- solution: Categorification!

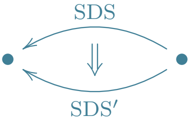
see Baez, Huerta, 1003.4485

We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance.
G. Moore and N. Seiberg, 1989

What does categorification mean?
One of Jeff Harvey's questions to identify the "generation PhD>1999" at Strings 2013.

- Consider self-dual strings:
  endpoints: objects
  string: morphisms of a category.

- Parallel transport along surface:
  morphism between morphisms



- This yields a 2-category: objects, 1-morphisms, 2-morphisms
- Nomenclature: 2-category $\equiv$ strict bicategory

- Most mathematical notions: Stuff endowed with Structure
- E.g.: Lie algebra: Vector space V with Lie bracket $[\cdot,\cdot]$:

$$[v,w]=-[w,v] \quad \text{and} \quad [u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0$$

- Internal categorification: (as opposed to: numbers $\rightarrow$ sets)
  - "stuff" $\rightarrow$ (small) category, objects and morphisms of "stuff"
  - "structure" $\rightarrow$ functors
  - structure relations hold "up to isomorphisms"
  - functors satisfy coherence axioms
- Weak Lie 2-algebra is a category $\mathcal{L}$: Roytenberg, 2007
  - objects and morphisms form vector spaces
  - endowed with functor $[\cdot,\cdot]:\mathcal{L}\times\mathcal{L}\rightarrow\mathcal{L}$
  - natural trafos: $\text{Alt}:[v,w]\Rightarrow-[w,v]$
    $\text{Jac}:[u,[v,w]]+[v,[w,u]]\Rightarrow-[w,[u,v]]$

Further Restrictions of Weak Lie 2-algebras:

- $\mathsf{Alt} = \mathrm{id}$: semistrict
- $\mathsf{Jac} = \mathrm{id}$: hemistrict
- $\mathsf{Alt} = \mathsf{Jac} = \mathrm{id}$: strict

Semistrict Lie n-algebras $\leftrightarrow$ n-term strong homotopy Lie algebras:

- Graded vector space/Complex:

$$L_{-n} \xrightarrow{\mu_1} \ldots \xrightarrow{\mu_1} L_1 \xrightarrow{\mu_1} L_0 \xrightarrow{\mu_1} 0$$

- Antisymmetric "products" $\mu_n : L^{\otimes n} \to L$ of degree $2-n$
- Higher/Homotopy Jacobi identities, e.g.

$$\mu_1^2 = 0 \ ,$$
$$\mu_1(\mu_2(\ell_1,\ell_2)) = \pm\mu_2(\mu_1(\ell_1),\ell_2) \pm \mu_2(\mu_1(\ell_2),\ell_2)$$
$$\mu_2(\mu_2(\ell_1,\ell_2),\ell_3) + \mathsf{cycl.} = \pm\mu_1(\mu_3(\ell_1,\ell_2,\ell_3))$$

- Known from: BV-quant., string FT, deformation quant., ...

Homotopy Maurer-Cartan equations (BV-quant., SFT)

Define curvatures. $F = \mathrm{d}A + \frac{1}{2}[A, A] = 0$ generalizes to

$$\mu_1(\phi) + \tfrac{1}{2}\mu_2(\phi,\phi) + \ldots = \sum_{i=1}^{\infty} \frac{(-1)^{i(i+1)/2}}{i!}\mu_i(\phi,\cdots,\phi) = 0$$

Gauge transformations $\delta A = \mathrm{d}\alpha + [A, \alpha]$ generalizes to

$$\delta\phi = \mu_1(\lambda) + \mu_2(\phi,\lambda) + \ldots = \sum_{i=1}^{\infty} \frac{(-1)^{i(i-1)/2}}{(i-1)!}\mu_i(\lambda,\phi,\cdots,\phi)$$

- Note: L_∞-algebra $\tilde{L} \to L = \Omega^\bullet(M) \otimes \tilde{L}$, degrees add.
- HMC equations for semistrict Lie 2-algebra:
  - $\phi = A + B \quad \in \quad L_1 \quad = \quad \Omega^1(M) \otimes \tilde{L}_0 \;\oplus\; \Omega^2(M) \otimes \tilde{L}_{-1}$
  - EOMs:
  $$\mathcal{F} = \mathrm{d}A + \tfrac{1}{2}\mu_2(A,A) - \mu_1(B) = 0$$
  $$\mathcal{H} = \mathrm{d}B + \mu_2(A,B) + \tfrac{1}{3!}\mu_3(A,A,A) = 0$$

Higher Gauge Theories

The most interesting higher gauge theories for us live in 6 and 4 dimensions.

- "Fake curvature": $\mathcal{F} = \mathrm{d}A + \frac{1}{2}\mu_2(A,A) - \mu_1(B) = 0$
  Vanishing makes parallel transport reparam. invariant.
  Rumour: $\mathcal{F} = 0 \Rightarrow$ theory abelian. This is false!
- 3-form curvature: $\mathcal{H} = \mathrm{d}B + \mu_2(A,B) + \frac{1}{3!}\mu_3(A,A,A) = 0$
  This describes a flat bundle, we can generalize this.

Gauge part of (2,0) theory

If (2,0) theory on $\mathbb{R}^{1,5}$ is a higher gauge theory, then gauge part is:

$$\mathcal{H} = *\mathcal{H}~, \quad \mathcal{F} = 0~.$$

Non-Abelian Self-Dual Strings

BPS equation for (2,0) theory on $\mathbb{R}^4$ ($\sim$ monopoles in 4d SYM)

$$\mathcal{H} = *\left(\mathrm{d}\Phi + \mu_2(A,\Phi)\right)~, \quad \mathcal{F} = 0~.$$

Later: solutions, categorified $\mathsf{SU}(2)$-Instanton/-monopole

Differential Lie Crossed Modules

Differential Lie crossed modules are strict Lie 2-algebras.

Restricting to $\mathsf{Alt} = \mathsf{Jac} = \mathrm{id}$ in a weak Lie 2-algebra yields:

Differential Lie crossed modules / Lie crossed modules

Pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, written as $(\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g})$ with:

- left automorphism action $\rhd\colon \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$
- group homomorphism $\mathsf{t} : \mathfrak{h} \to \mathfrak{g}$

$$\mathsf{t}(g \rhd h) = [g, \mathsf{t}(h)] \quad \text{and} \quad \mathsf{t}(h_1) \rhd h_2 = [h_1, h_2]$$

- Finite version: Lie crossed module $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$

Simplest examples:

- Lie group G, Lie crossed module: $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$.
- Abelian Lie group G, Lie crossed module: $\mathsf{BG} = (\mathsf{G} \xrightarrow{\mathsf{t}} 1)$.

More involved:

- Automorphism 2-group of Lie group G: $(\mathsf{G} \xrightarrow{\mathsf{t}} \mathsf{Aut}(\mathsf{G}))$

Principal 2-Bundles with Connective Structures

Higher gauge theory is the dynamical theory of principal 2-bundles.

Consider a manifold M with cover (U_a)

Object	Principal G-bundle	Principal $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle
Cochains	(g_{ab}) valued in G	(g_{ab}) valued in G, (h_{abc}) valued in H
Cocycle	$g_{ab}g_{bc} = g_{ac}$	$\mathsf{t}(h_{abc})g_{ab}g_{bc} = g_{ac}$ $h_{acd}h_{abc} = h_{abd}(g_{ab} \rhd h_{bcd})$
Coboundary	$g_a g'_{ab} = g_{ab}g_b$	$g_a g'_{ab} = \mathsf{t}(h_{ab})g_{ab}g_b$ $h_{ac}h_{abc} = (g_a \rhd h'_{abc})h_{ab}(g_{ab} \rhd h_{bc})$
gauge pot.	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$, $B_a \in \Omega^2(U_a) \otimes \mathfrak{h}$
Curvature	$F_a = \mathrm{d}A_a + A_a \wedge A_a$	$\mathcal{F}_a = \mathrm{d}A_a + A_a \wedge A_a - \mathsf{t}(B_a) \overset{!}{=} 0$ $\mathcal{H}_a = \mathrm{d}B_a + A_a \rhd B_a$
Gauge trafos	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a$	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a + \mathsf{t}(\Lambda_a)$ $\tilde{B}_a := g_a^{-1} \rhd B_a + \tilde{A}_a \rhd \Lambda_a + \mathrm{d}\Lambda_a - \Lambda_a \wedge \Lambda_a$

Remarks:

- A principal $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle is a principal G-bundle.
- A principal $(\mathsf{U}(1) \xrightarrow{\mathsf{t}} 1) = \mathsf{BU}(1)$-bundle is an abelian gerbe.
- Gauge part of (2,0) theory even clear for non-trivial M.

Application:

Constructing Superconformal (2,0) Theories using Twistor Spaces

Twistor Description of Higher Yang-Mills Fields

Using twistor spaces, one can map holomorphic data to solutions to field equations.

Details $\Rightarrow$ Martin Wolf's talk later

Recall the principle of the Penrose-Ward transform:

- We construct a double fibration

$$\begin{array}{ccc} & F & \\ \swarrow & & \searrow \\ P & & M \end{array}$$

  P: twistor space, F: correspondence space
- $H^n(P,\mathfrak{S})$ (e.g. vector bundles) $\stackrel{1:1}{\longleftrightarrow}$ sols. to field equations.
- Our new contributions:
  - Use non-abelian gerbes
  - New twistor space
- Can describe in this way:
  - 6d (2,0) superconformal equations of motion
  - self-dual strings

Context:

The ABJM Model as a Higher Gauge Theory

The ABJM Model as a Higher Gauge Theory

The ABJM model can be completed to a higher gauge theory.

- Most dualities in string theory between Yang-Mills theories.
- And in M-theory? M2-branes: Chern-Simons-matter theories
  M5-branes: Tensor-multiplet theories
- These can be put on equal footing. S Palmer&CS, 1311.1997

Step 1: The ABJM gauge structures / hermitian 3-Lie algebras

- form differential crossed modules. S Palmer&CS, 1203.5757
- but: $\mathsf{t} = 0$, thus $F = \mathsf{t}(B) = 0$.
- Recall: Lie algebra $\mathfrak{g} \rightarrow$ inner derivation dcm $\mathfrak{g} \xrightarrow{\mathsf{t}} \mathfrak{g}$
- dcm $\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g} \rightarrow$ inner derivation d2-cm $\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g} \ltimes \mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g}$

Explicitly:

$$\begin{pmatrix} 0 & \mathfrak{gl}(N,\mathbb{C}) \\ 0 & 0 \end{pmatrix} \xrightarrow{\mathsf{t}} \begin{pmatrix} \mathfrak{u}(N) & \mathfrak{gl}(N,\mathbb{C}) \\ 0 & \mathfrak{u}(N) \end{pmatrix} \xrightarrow{\mathsf{t}} \begin{pmatrix} \mathfrak{u}(N) & 0 \\ 0 & \mathfrak{u}(N) \end{pmatrix}$$

Step 2: Implement the fake curvature conditions

- Here, we are working with a differential 2-crossed module.
- Gauge potentials: A, B, C. Curvatures: F, H, G.
- Conditions $\mathcal{F} = F - \mathsf{t}(B) = 0$, $\mathcal{H} = H - \mathsf{t}(C) = 0$
- Action:

$$S_{\rm ABJM} = \int_{\mathbb{R}^{1,2}} {\rm tr}\Big(\tfrac{k}{4\pi}\eta\ A\wedge({\rm d}A + \tfrac{1}{3}[A,A]) - \nabla Z^\dagger_A \wedge *\nabla Z^A - *{\rm i}\bar{\psi}^A \wedge \nabla\!\!\!/\,\psi_A\Big) + V$$

$$S_{\rm HGT} = S_{\rm ABJM} + \int_{\mathbb{R}^{1,2}} {\rm tr}\Big(\lambda_1^\dagger \wedge (F - \mathsf{t}(B)) + \lambda_2^\dagger(H - \mathsf{t}(C)) + \lambda_3^\dagger \mathsf{t}(\lambda_2)\Big)$$

- This yields ABJM eoms + fake curvature constraints

Application:

Higher Monopole and Instanton Solutions

Review: The BPST Instanton

The BPST instanton can be conveniently written using quaternions.

Recall the quaternionic form of the elementary instanton on S^4:

Conformal geometry of S^4

Describe S^4 by $\mathbb{H} \cup \{\infty\}$. Coordinates: $x = x^1 + \mathrm{i}x^2 + \mathrm{j}x^3 + \mathrm{k}x^4$.
Conformal transformations:

$$x \mapsto (ax+b)(cx+d)^{-1}~, \quad a,b,c,d \in \mathbb{H}$$

SU(2)-Instanton:

$$A = \mathrm{im}\left(\frac{\bar{x}\mathrm{d}x}{1+|x|^2}\right) \quad \Rightarrow \quad F = \mathrm{im}\left(\frac{\mathrm{d}\bar{x}\wedge\mathrm{d}x}{(1+|x|^2)^2}\right)$$

SU(2)-Anti-Instanton:

$$A = \mathrm{im}\left(\frac{x\mathrm{d}\bar{x}}{1+|x|^2}\right) \quad \Rightarrow \quad F = \mathrm{im}\left(\frac{\mathrm{d}x\wedge\mathrm{d}\bar{x}}{(1+|x|^2)^2}\right)$$

Belavin et al. 1975, Atiyah 1979

Elementary Solution: The Higher Instanton

The quaternionic form of the BPST instanton solution translates perfectly.

Solution to the higher instanton equations $H = \star H$, $F = \mathsf{t}(B)$:

- Same inner derivation 2-crossed module as for ABJM
- Recall BPST instanton:

$$A = \mathrm{im}\left(\frac{\bar{x}\mathrm{d}x}{1+|x|^2}\right) \quad \Rightarrow \quad F = \mathrm{im}\left(\frac{\mathrm{d}\bar{x}\wedge\mathrm{d}x}{(1+|x|^2)^2}\right)$$

- Solution in coordinates $x = x^M\sigma_M$, $\hat{x} = x^M\bar{\sigma}_M$

$$A = \begin{pmatrix} \frac{\hat{x}\,\mathrm{d}x}{1+|x|^2} & 0 \\ 0 & \frac{\mathrm{d}x\,\hat{x}}{1+|x|^2} \end{pmatrix} \quad B = F + \begin{pmatrix} 0 & \frac{\hat{x}\,\mathrm{d}x\wedge\mathrm{d}\hat{x}}{(1+|x|^2)^2} \\ 0 & 0 \end{pmatrix}$$

$$F := \mathrm{d}A + A\wedge A = \begin{pmatrix} \frac{\mathrm{d}\hat{x}\wedge\mathrm{d}x}{(1+|x|^2)^2} + \frac{2\,\mathrm{d}\hat{x}\,x\wedge\mathrm{d}\hat{x}\,x}{(1+|x|^2)^2} & 0 \\ 0 & -\frac{\mathrm{d}x\wedge\mathrm{d}\hat{x}}{(1+|x|^2)^2} \end{pmatrix}$$

$$H := \mathrm{d}B + A \rhd B = \begin{pmatrix} 0 & \frac{\mathrm{d}\hat{x}\wedge\mathrm{d}x\wedge\mathrm{d}\hat{x}}{(1+|x|^2)^3} \\ 0 & 0 \end{pmatrix}$$

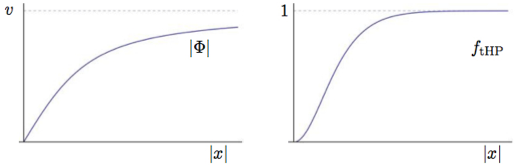
Review: The 't Hooft-Polyakov Monopole

The 't Hooft-Polyakov Monopole is a non-singular solution with charge 1.

Recall 't Hooft-Polyakov monopole (e_i generate $\mathfrak{su}(2)$, $\xi = v|x|$):

$$\Phi = \frac{e_i x^i}{|x|^2}(\xi \coth(\xi) - 1)\ , \quad A = \varepsilon_{ijk}\frac{e_i x^j}{|x|^2}\left(1 - \frac{\xi}{\sinh(\xi)}\right)\ \mathrm{d}x^k$$

- At S_2^∞: $\Phi \sim g(\theta) e_3 g(\theta)^1$.
  $g(\theta) : S^2_\infty \to \mathsf{SU}(2)/\mathsf{U}(1)$: winding 1
- Charge $q = 1$ with
  $$2\pi q = \tfrac{1}{2}\int_{S^2_\infty} \frac{\mathrm{tr}\,(F^\dagger \Phi)}{||\Phi||} \quad \text{with} \quad ||\Phi|| := \sqrt{\tfrac{1}{2}\,\mathrm{tr}\,(\Phi^\dagger \Phi)}$$
- Higgs field non-singular:



Elementary Solutions: A Non-Abelian Self-Dual String

We can write down a non-abelian self-dual string with winding number 1.

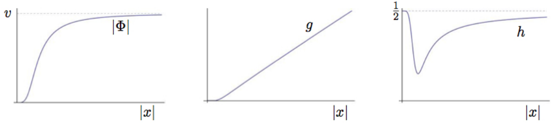
Self-Dual String (e_μ generate DCM $\mathfrak{su}(2)\times\mathfrak{su}(2) \xrightarrow{\mathsf{t}} \mathbb{R}^4$, $\xi = v|x|^2$):

$$\Phi = \frac{e_\mu x^\mu}{|x|^3}\, f(\xi)\ , \quad B_{\mu\nu} = \varepsilon_{\mu\nu\kappa\lambda}\frac{e_\kappa x^\lambda}{|x|^3}\, g(\xi)\ , \quad A_\mu = \varepsilon_{\mu\nu\kappa\lambda} D(e_\nu, e_\kappa)\, \frac{x^\lambda}{|x|^2}\, h(\xi)$$

- At S^∞_3: $\Phi \sim g(\theta) \triangleright e_4$. $g(\theta) : S^3_\infty \to \mathsf{SU}(2)$ has winding 1.
- Charge $q = 1$:

$$(2\pi)^3 q = \tfrac{1}{2}\int_{S^3_\infty} \frac{(H,\Phi)}{||\Phi||} \quad \text{with} \quad ||\Phi|| := \sqrt{\tfrac{1}{2}(\Phi,\Phi)}\ ,$$

- Higgs field non-singular:



What I didn't have time to talk about...

There is much more evidence for using higher structures in M-theory.

- 6d (1,0) models from tensor hierarchies
  Samtleben et al., 1108.4060, also 1108.5131
  - (1,0) tensor + vector multiplets with new gauge structure
  - These are higher gauge theories.
  - New gauge structure: symplectic Lie n-algebroids
    S Palmer&CS 1308.2622, Samtleben et al. 1403.7114
- Geometric Quantization (Noncommutative/Fuzzy spaces)
  - Analogues by quantizing "Poisson Lie 2-algebras"
  - This yields nonassociative geometry.
  - A categorified IKKT model can be written down.
  - This model has nonassociative geometry solutions.
  - Background expansion: nonassociative HGT
    P Ritter&CS 1308.4892
- HGT a very nice playground, particularly for PhD students:
  - Higher Magnetic Bags S Palmer&CS 1204.6685
  - Proof of Higher Poincaré Lemma G Demessie&CS 1406.5342

Summary:

- ✓ Clear physical and mathematical motivation to study HGT
- ✓ Generalized ADHMN-like construction on loop space
- ✓ Various twistor constructions with non-abelian gerbes
- ✓ 6d superconformal tensor multiplet equations
- ✓ (1,0) models of Samtleben et al. is HGT
- ✓ ABJM model is a HGT
- ✓ Explicit higher monopole and instanton solutions

Future directions:

- ▷ Twistor spaces of loop spaces
- ▷ Continue translation of higher ADHM-constructions
- ▷ Geometric Quant. with higher Hilbert spaces
- ▷ Study categorified IKKT model

Higher Gauge Theory and M-theory

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