G_2–instantons over twisted connected sums

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Gauge theory in higher dimensions

E.g.: Riemannian manifold (X, g) of dimension $n \geq 4$,
$SU(m)-$bundle $E \to X$, A connection on E.

The *Yang-Mills functional*:

$$YM(A) \doteq \|F_A\|^2 = \int_X \langle F_A \wedge *F_A \rangle_{\mathfrak{su}(m)},$$

induces the (Euler-Lagrange) *Yang-Mills equation*

$$d_A^* F_A = 0.$$

$n = 4 : \quad \Omega^2 = \Omega^2_+ \oplus \Omega^2_-, \quad F_A = \pm * F_A$ (*SD* or *ASD*) sols.
$n > 4 : \quad$?

Tian et al.: a *closed* $(n-4)-$form Θ on X generalises (A)SD:

$$F_A \wedge \Theta = - * F_A \qquad [\Theta - \text{instanton}].$$

¿How to find closed tensors?

Holonomy theorem:

$$\exists S \in \Gamma(\mathcal{T}) \quad \text{s.t.} \quad \nabla S = 0$$
$$\Updownarrow$$
$$\exists x \in X, S_x \in \mathcal{T}_x \quad \text{s.t.} \quad \text{Hol}(g).S_x = S_x$$

with $\mathcal{T} \doteq (\bigotimes^{\bullet} TX \otimes \bigotimes^{\bullet} T^*X)$.

Berger's list

(M^n, g) simply-connected Riemannian manifold, g irreducible and nonsymmetric; then exactly one of the following holds:

1. Hol $(g) = SO(n)$
2. $n = 2m, m \geq 2$: Hol $(g) = U(m) \subset SO(2m)$
3. $\mathbf{n = 2m, m \geq 2}$: $\boxed{\text{Hol } (\mathbf{g}) = \mathbf{SU}(\mathbf{m}) \subset \mathbf{SO}(\mathbf{2m})}$
4. $n = 4m, m \geq 2$: Hol $(g) = Sp(m) \subset SO(4m)$
5. $n = 4m, m \geq 2$: Hol $(g) = Sp(m)\,Sp(1) \subset SO(4m)$
6. $\mathbf{n = 7, m \geq 2}$: $\boxed{\text{Hol } (\mathbf{g}) = \mathbf{G_2} \subset \mathbf{SO}(\mathbf{7})}$
7. $n = 8, m \geq 2$: Hol $(g) =$ Spin $(7) \subset SO(8)$.

We will be interested in the interplay between the following instances:

Hol $(g) = SU(3)$	(Calabi-Yau $3-$folds):	$\omega^{1,1}, \Omega^{3,0}$
Hol $(g) \subseteq G_2$	(G_2-manifolds):	φ^3 (and $*\varphi^4$)

$$\varphi_{\mathbf{0}} = \left(\mathbf{e^{12}} - \mathbf{e^{34}}\right)\mathbf{e^{5}} + \left(\mathbf{e^{13}} - \mathbf{e^{42}}\right)\mathbf{e^{6}} + \left(\mathbf{e^{14}} - \mathbf{e^{23}}\right)\mathbf{e^{7}} + \mathbf{e^{567}}$$

$\{e^i\}_{i=1,\dots,7}$ canonical basis of $\left(\mathbb{R}^7\right)^*$, $e^{ij} = e^i e^j \doteq e^i \wedge e^j$ etc.

$$\boxed{G_2 \doteq \{g \in GL(7) \mid g^*\varphi_0 = \varphi_0\}}$$

A G_2-*structure* on M^7 is a form $\varphi \in \Omega^3(M)$ s.t.,

$$\varphi_p = f_p^*(\varphi_0)$$

for some frame $f_p : T_pM \to \mathbb{R}^7, \quad \forall p \in M$.

If $\nabla\varphi = 0$ (torsion-free), $\left(M^7, \varphi\right)$ is a G_2-*manifold*; then we have

$$d\varphi = 0, \quad d *_\varphi \varphi = 0 \quad \text{and} \quad \text{Hol}(\varphi) \subseteq G_2.$$

(W,ω) Kähler manifold, $\mathcal{E} \to W$ holomorphic vector bundle:

$$\left\{ \begin{array}{c} \text{Hermitian metrics} \\ H \text{ on } \mathcal{E} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{unitary (Chern) connections} \\ A = A_H \text{ on } \mathcal{E} \end{array} \right\};$$

in particular, $F_{A_H} \in \Omega^{1,1}(\mathfrak{g})$. Then H is *Hermitian Yang-Mills (HYM)* if the curvature has vanishing $\omega-$trace:

$$\hat{F}_A \doteq (F_A, \omega) = 0.$$

Proposition. A HYM connection A on a hol. v.b. $\mathcal{E} \to W$ over a CY $3-$fold W lifts to a G_2-instanton on $p_1^*\mathcal{E} \to M = W \times S^1$, where

$$\begin{aligned} \varphi &= \omega \wedge d\theta + \mathrm{Im}\,\Omega, \\ *\varphi &= \tfrac{1}{2}\omega \wedge \omega - \mathrm{Re}\,\Omega \wedge d\theta. \end{aligned}$$

Twisted connected sums

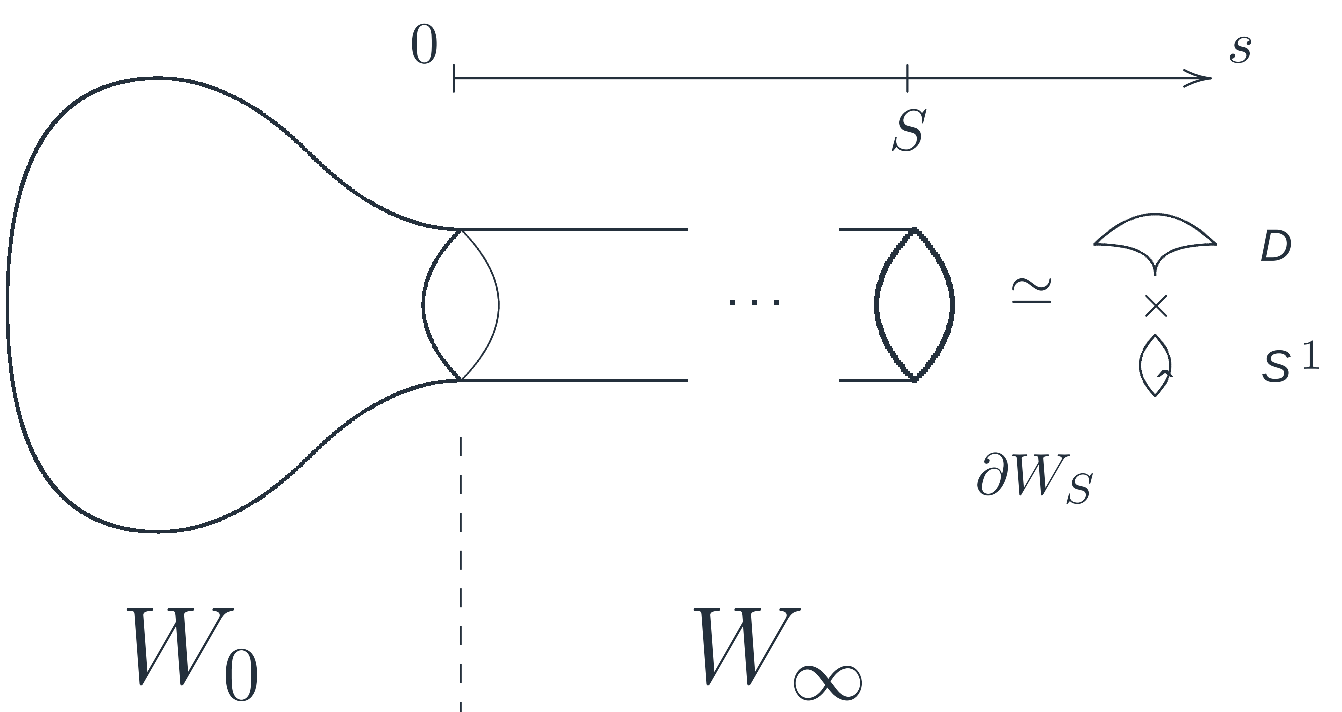
Ingredients

$(X^3, \bar{\omega}, I)$ compact, simply-connected, Kähler with:

- $\exists\, K3-$surface $D \in |-K_X|$ with $\mathcal{N}_{D/X}$ (hol.) trivial;
- The complement $W = X \setminus D$ has finite $\pi_1(W)$.

Think of W as $W = W_0 \cup W_\infty$, where W_0 is compact with boundary and

$$\partial W_0 \simeq D \times S^1, \quad W_\infty \simeq \left(D \times S^1 \times \mathbb{R}_+\right).$$



Theorem (Calabi-Yau-Tian-Kovalev-CHNP). For $W = X \setminus D$ as above:

1. W admits a complete Ricci-flat Kähler structure ω;
2. Hol $(\omega) = SU(3)$, i.e. W is Calabi-Yau;
3. along the tubular end $D \times S^1_\alpha \times (\mathbb{R}_+)_s$, the Kähler form ω and the holomorphic volume form Ω are exponentially asymptotic[1] to those of the product Ricci-flat Kähler metric on D:

$$\begin{aligned} \omega|_{W_\infty} &= \kappa_I + ds \wedge d\alpha + d\psi \\ \Omega|_{W_\infty} &= (ds + \mathbf{i} d\alpha) \wedge (\kappa_J + \mathbf{i}\kappa_K) + d\Psi. \end{aligned}$$

We say (W, ω) is an *(exponentially) asymptotically cylindrical Calabi-Yau (ACylCY)* $3-$fold.
NB.: κ_I, κ_J and κ_K (hyper-)Kähler forms on the K3 surface D.

[1] with d$\psi, d\Psi = O\left(e^{-s}\right)$.

A *building block* is a nonsingular algebraic $3-$fold X with projective morphism $f : X \to \mathbb{P}^1$ such that $D := f^*(\infty) \in |-K_X|$ is a nonsingular K3 surface $(\ldots)$. Building blocks admit ACylCY metrics.

Matching data for a pair of building blocks $(X_\pm, D_\pm)$:

$$\mathbf{m} = \{(\omega_{I,\pm}, \omega_{J,\pm}, \omega_{K,\pm}), \mathfrak{r}\}$$

- choice of hyperkähler structures on $D_\pm$ such that $[\omega_{I,\pm}] = [\bar{\omega}|_{D_\pm}]$,
- hyperkähler rotation $\mathfrak{r} : \ D_+ \to D_-$, i.e., diffeo $\mathfrak{r} : \ D_+ \to D_-$ s.t.

$$\mathfrak{r}^*\omega_{I,-} = \omega_{J,+}, \quad \mathfrak{r}^*\omega_{J,-} = \omega_{I,+} \quad \text{and} \quad \mathfrak{r}^*\omega_{K,-} = -\omega_{K,+}.$$

These can be obtained as $X := \mathrm{Bl}_C V$ for e.g. (weak) Fanos:

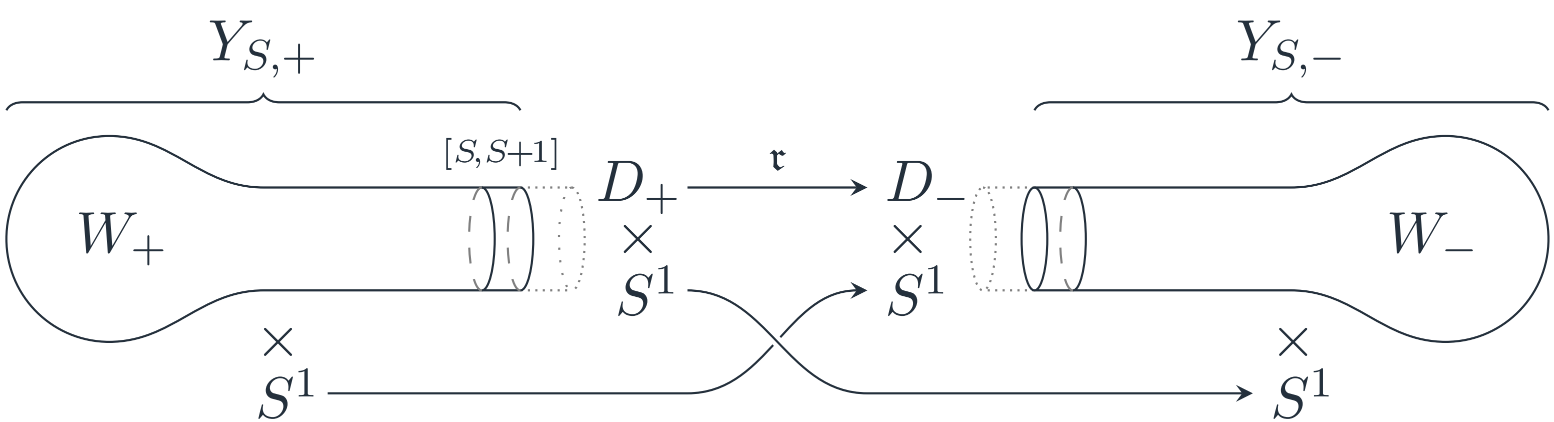
- $V = \mathbb{P}^3$
- $V \subset \mathbb{P}^4, \deg(V) = 2, 3$
- $V = \mathbb{P}^2 \times \mathbb{P}^1$
- $V_{22} \hookrightarrow \mathbb{P}^{13} \quad (g = 12)$:-)

Given a suitable pair of 3-folds W_+ and W_- as above, one obtains a compact oriented $7-$manifold

$$Y_S = \left(W_{S,+} \times S^1\right) \cup_{\mathfrak{r}} \left(W_{S_-} \times S^1\right) =: W_+ \widetilde{\#}_S W_- .$$

'Stretching the neck', one equips Y_S with a G_2-structure φ_S satisfying exactly

$$\text{Hol}\left(\varphi_S\right) = \mathrm{G}_2 .$$



The Hermitian Yang-Mils problem

Goal: to solve the HYM problem $\hat{F}_H = 0$ on suitable bundles over these ACylCYs, *ergo* G_2-instantons on the pull-back over $W \times S^1$.
Strategy: consider first the 'nonlinear heat flow'

$$(\dagger) \begin{cases} H^{-1}\dfrac{\partial H}{\partial t} &= -2\mathbf{i}\hat{F}_H \\ H(0) &= H_0 \end{cases} \quad \text{on} \quad W_S \times [0,T[$$

over a truncation, with (Dirichlet) boundary condition

$$H\,|_{\partial W_S} = H_0\,|_{\partial W_S}$$

where H_0 is a fixed *reference (Hermitian) metric* on $\mathcal{E} \to W$ with 'good' asymptotic behaviour. Then

$$H = \lim_{t<T\to\infty}\left(\lim_{S\to\infty} H_S(t)\right).$$

Definition. A bundle $\mathcal{E} \to W$ is *stable at infinity* (or *asymptotically stable*) if it is the restriction of a hol.v.b. $\mathcal{E} \to X$ satisfying:

- $\mathcal{E}$ is indecomposable;
- $\mathcal{E}|_D$ is stable, hence also $\mathcal{E}|_{D_z}$ for $|z| < \delta$.

Definition. A *reference metric* H_0 on an asymptotically stable bundle $\mathcal{E} \to W$ is (the restriction of) a smooth Hermitian metric on $\mathcal{E} \to X$ such that:

- $H_0|_{D_z}$ are the HYM metrics on $\mathcal{E}|_{D_z}$, $0 \leq |z| < \delta$;
- H_0 has finite energy: $\|\hat{F}_{H_0}\|_{L^2(W,\omega)} < \infty$.

Following S. Donaldson, C. Simpson et al.:

- Each $H_S(t)$ exists and is unique and smooth, $\forall t \in \left]0,\infty\right[$.
- $H_S(t)$ are bounded in $L^p_2(W_S)$ uniformly in t, $\forall 1 \leq p < \infty$.
- Consequently, $H_S(T)$ is of class C^1 [Sobolev embedding] and $\|F_{H_S}\|_{L^p(W_S)} < \infty, \quad \forall 1 \leq p < \infty$.
- $(\ldots)$ *gruesome analysis* (...)
- F_{H_S} is actually bounded in $L^\infty_k(W_S)$ [bounds on Heat Kernel].
- $H(T)$ is smooth [elliptic regularity].

Proposition. *Given any $T > 0$, $\exists ! \{H(t)\}$ on $\mathcal{E} \to W$, solution of the evolution equations*

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2\mathbf{i}\hat{F}_H \\ H(0) = H_0 \end{cases} \quad \textit{on} \quad W \times [0,T]$$

with 'good' asymptotic behaviour.

Theorem 1 (S., 2011). *Let $\mathcal{E} \to W$ be asymptotically stable, equipped with a reference metric H_0, over an ACylCY 3–fold W as given by the C-Y-T-K-CHNP theorem, and let $\{H_t\}_{t\in]0,\infty[}$ be the 1−parameter family of Hermitian metrics on $\mathcal{E}$ solving the evolution equation $(\dagger)$ over W;*

then the limit $H = \lim_{t\to\infty} H_t$ exists and is a smooth HYM metric on $\mathcal{E}$, exponentially asymptotic in all derivatives to H_0 along the tubular end:

$$\boxed{\hat{F}_H = 0, \quad H \xrightarrow[S\to\infty]{C^\infty} H_0.}$$

...

so we have a G_2-instanton on $p_1^*\mathcal{E} \to W \times S^1$!!!

Theorem 2 (S.-Walpuski, 2013). $(X_\pm, D_\pm, \mathbf{m})$ *matching pair of building blocks,* $(Y, \varphi_S) := X_+ \tilde{\#}_S X_-$ *the compact 7–manifold and* $\mathcal{E}_\pm \to X_\pm$ *holomorphic bundles s.t.*

- *Stability:* $\mathcal{E}_\pm|_{D_\pm}$ *is stable with corresponding ASD instanton* $A_{\infty,\pm}$.
- *Compatibility: isomorphism* $\bar{\mathfrak{r}} \colon \mathcal{E}_+|_{D_+} \to \mathcal{E}_-|_{D_-}$ *covering* $\mathfrak{r}$ *s.t.* $\bar{\mathfrak{r}}^* A_{\infty,-} = A_{\infty,+}$.
- *Rigidity: no infinitesimal deformations of* $\mathcal{E}_\pm$ *fixing restriction to* $D_\pm$:

$$H^1(X_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm)(-D_\pm)) = 0.$$

- *Transversality:* $\mathrm{im}(\lambda_+) \cap \mathrm{im}(\bar{\mathfrak{r}}^* \circ \lambda_-) = \{0\} \subset H^1_{A_{\infty,+}}$ *for*

$$\lambda_\pm \colon H^1(X_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm)) \to H^1_{A_{\infty,\pm}} := \ker\left(\mathrm{d}^*_{A_{\infty,\pm}} \oplus \mathrm{d}^+_{A_{\infty,\pm}}\right)\Big|_{D_\pm}$$

Then there exists a non-trivial $\mathbf{PU}(n)$*–bundle* E *over* Y, *a constant* $S_1 \geq S_0$ *and for each* $S \geq S_1$ *an irreducible (...)* G_2*–instanton* A_S *on* E *over* (Y, φ_S).

Construction of asymptotically stable bundles

Recall the *slope* of a coherent sheaf is

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\mathrm{rk}\,(\mathcal{F})}.$$

A holomorphic vector bundle $\mathcal{E} \to X$ (think $X = X$) is said to be *stable* if, for every coherent subsheaf $\mathcal{F} \hookrightarrow \mathcal{E}$,

$$\mu\,(\mathcal{F}) < \mu(\mathcal{E}).$$

A smooth projective variety X is *cyclic* if Pic $(X) = \mathbb{Z}$; then

$$\deg(\mathcal{E}) := c_1(\mathcal{E}) \cdot \mathcal{O}_X(1)^{\otimes(\dim X - 1)}$$

The *normalisation* of $\mathcal{E} \to X$ is $\mathcal{E}_{\text{norm}} := \mathcal{E}(-k_{\mathcal{E}})$, with $k_{\mathcal{E}} := \lceil \mu(\mathcal{E}) \rceil \in \mathbb{Z}$. Clearly

$$-r + 1 \leq c_1(\mathcal{E}(-k_{\mathcal{E}})) = c_1(\mathcal{E}) - r.k_{\mathcal{E}} \leq 0.$$

Criterion (Hoppe). *Let $\mathcal{E} \to \mathbb{P}^n$ be a holomorphic vector bundle of rank $r = 2$ and $c_1(\mathcal{E}) = 0$; then*

$$\mathcal{E} \text{ is stable } \Leftrightarrow h^0(\mathcal{E}) = 0.$$

Proof.
($\Rightarrow$) A section $s \in H^0(\mathcal{E})$ would give a monomorphism $\mathcal{O}_{\mathbb{P}^n} \hookrightarrow \mathcal{E}$, violating stability: $\mu(\mathcal{O}_{\mathbb{P}^n}) = 0 \geq 0 = \mu(\mathcal{E})$.
($\Leftarrow$) A destabilising sheaf $\mathcal{F} \hookrightarrow \mathcal{E}$ must be a line bundle.
Since Pic $(\mathbb{P}^n) = \mathbb{Z}$, we have $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n}(a)$ whose inclusion is a section of $\mathcal{E}(-a)$. By assumption, we must have $a < 0$, but then

$$a = \mu(\mathcal{F}) \geq \mu(\mathcal{E}) = 0 > a \quad (!) \quad \square$$

¿What about arbitrary degree?

Criterion (Hoppe II). Let $\mathcal{E} \to X$ be a holomorphic vector bundle of rank $r = 2$ over a cyclic variety; then

$$\mathcal{E} \text{ is stable } \Leftrightarrow h^0\left(\mathcal{E}_{\text{norm}}\right) = 0.$$

Proof.
$(\Rightarrow)$ A section $s \in H^0(\mathcal{E}_{\text{norm}})$ would give $\mathcal{O}_X(k_{\mathcal{E}}) \hookrightarrow \mathcal{E}$, violating stability: $\mu(\mathcal{O}_X(k_{\mathcal{E}})) = k_{\mathcal{E}} \geq \frac{c_1(\mathcal{E})}{2} = \mu(\mathcal{E})$.
$(\Leftarrow)$ A destabilising sheaf $\mathcal{F} \hookrightarrow \mathcal{E}$ must be a line bundle.
Since Pic$(X) = \mathbb{Z}$, $\mathcal{F} = \mathcal{O}_X(a)$ which gives a section of $\mathcal{E}(-a) = \mathcal{E}_{\text{norm}}(k_{\mathcal{E}} - a)$. By hypothesis, we must have $a < k_{\mathcal{E}}$, but then

$$a = \mu(\mathcal{F}) \geq \mu(\mathcal{E}) > k_{\mathcal{E}} - 1 \geq a \quad (!) \quad \square$$

¿What about arbitrary rank $r \geq 2$?

Criterion (Hoppe III). Let $\mathcal{E} \to X$ be a holomorphic vector bundle of rank r over a cyclic variety; if

$$H^0((\wedge^s \mathcal{E})_{\mathrm{norm}}) = 0 \quad \text{for} \quad 1 \leq s \leq r-1,$$

then $\mathcal{E}$ is stable. Conversely, if $\mathcal{E}$ is stable then $H^0(\mathcal{E}_{\text{norm}}) = 0$.

Proof.
($\Rightarrow$) A section of $\mathcal{E}_{\text{norm}}$ would give $\mathcal{O}(k_{\mathcal{E}}) \hookrightarrow \mathcal{E}$, violating stability: $\mu(\mathcal{O}(k_{\mathcal{E}})) = k_{\mathcal{E}} \geq \frac{c_1(\mathcal{E})}{r} = \mu(\mathcal{E})$.
($\Leftarrow$) A destabilising $\mathcal{F} \hookrightarrow \mathcal{E}$ of rank s gives $\wedge^s \mathcal{F} \hookrightarrow \wedge^s G$, hence a section of $(\wedge^s \mathcal{E})(-a)$ with $\det \mathcal{F} = \mathcal{O}_X(a)$.
By hypothesis, we must have $a < k_s := k_{\wedge^s \mathcal{E}}$, but then

$$a = \deg \mathcal{F} \geq s.\mu(\mathcal{E}) = \mu(\wedge^s \mathcal{E}) > k_s - 1 \geq a \quad (!) \quad \square$$

Proposition 3 (Jardim-S.). *Let $\mathcal{E} \to X^3$ be the vector bundle, over a nonsingular cyclic Fano variety, arising from an instanton monad of the form*

$$0 \longrightarrow \mathcal{O}(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}(1)^{\oplus c} \longrightarrow 0 \qquad (1)$$

Then $\mathcal{E} := \dfrac{\ker \beta}{\operatorname{img} \alpha}$ is stable. If moreover $D \subset X$ is a cyclic divisor, then $\mathcal{E}\big|_D$ is stable.

Recall: Therefore $\mathcal{E}$ is a G_2–instanton bundle, by Theorem 1.

Trivial example: null-correlation bundle over $\mathbb{P}^3$

$$X = \mathbb{P}^3, \qquad D \in |-K_{\mathbb{P}^3}| = |\mathcal{O}(4)|, \qquad r = 2, \quad c = 1$$

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \mathcal{O}(-5) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & K(-4) & \longrightarrow & \mathcal{O}(-4)^{\oplus 4} & \longrightarrow & \mathcal{O}(-3) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E}(-4) & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}|_D & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

¿How about larger Picard group, e.g. $\mathbb{P}^2 \times \mathbb{P}^1$?

A polycyclic Hoppe theory

A projective variety X will be called *polycyclic* if Pic $(X) \simeq \mathbb{Z}^{l+1}$ for some $l \geq 0$. (e.g., all weak Fano 3-folds).
Fix Pic $(X) = \langle \Upsilon_0, \Upsilon_1, \ldots, \Upsilon_l \rangle$; given $\vec{p} \in \mathbb{Z}^{l+1}$ one denotes

$$\mathcal{O}(\vec{p}) = \mathcal{O}_X(p_0, \ldots, p_l) := \Upsilon_0^{\otimes p_0} \otimes \cdots \otimes \Upsilon_l^{\otimes p_l}.$$

Accordingly, given $\mathcal{E} \to X$, its *polytwist* is denoted by

$$\mathcal{E}(\vec{p}) = \mathcal{E}(p_0, \ldots, p_l) := \mathcal{E} \otimes \mathcal{O}(p_0, \ldots, p_l).$$

Set $[h_i] := c_1(\Upsilon_i) \in H^2(X, \mathbb{Z})$. For a torsion-free coherent sheaf $\mathcal{F}$ of rank s and $[c_1(\mathcal{F})] = p_0[h_0] + \cdots + p_l[h_l]$:

$$\det \mathcal{F} = (\wedge^s \mathcal{F})^{\vee\vee} = \mathcal{O}(p_0, \ldots, p_l),$$

L-degree

Fix polarisation $L \to X$; the $L-$*degree of* $\mathcal{F}$ is

$$\deg_L \mathcal{F} := c_1(\mathcal{F}) \cdot L^{\dim X - 1}$$

and it induces a linear functional δ_L on the lattice $\mathbb{Z}^{l+1}$:

$$\delta_L(p_0, \ldots, p_l) := \deg_L \mathcal{O}(p_0, \ldots, p_l).$$

Denoting by $\{\overrightarrow{e_i}\}_{i=0,\cdots,l}$ the canonical basis of $\mathbb{Z}^{l+1}$:

$$\deg_L \mathcal{F}(m\overrightarrow{e_i}) = \deg_L \mathcal{F} + m\,(\text{rank}\,\mathcal{F})\,\delta_L(\overrightarrow{e_i}).$$

Example 1 (Cartesian product). $X = \mathbb{P}^n \times \mathbb{P}^m$, $L := \mathcal{O}(1,1)$:

$$\deg_L \mathcal{F} = \frac{n(n+1)\cdots(n+m-1)}{m!}\left(p_1 + \frac{m}{n}p_2\right).$$

Example 2 (Hirzebruch surfaces). $X = \Sigma_a := \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}\right)$, Pic$\,(X) = \mathbb{Z}.S_a \oplus \mathbb{Z}.H$, $L := \mathcal{O}(1, a+1)$; *if* $\det \mathcal{F} = \mathcal{O}(p_1, p_2)$, *then*

$$\deg_L \mathcal{F} = (a+1)p_1 + p_2 - ap_1 = p_1 + p_2.$$

Example 3 (Blow-up of $\mathbb{P}^2$ at l points). $X = \tilde{\mathbb{P}}^2(l)$, Pic$\,(X) = \mathbb{Z}.E_1 \oplus \cdots \oplus \mathbb{Z}.E_l \oplus \mathbb{Z}.H$, $L := \mathcal{O}(-1,\ldots,-1,l+1)$; *if* $\det \mathcal{F} = \mathcal{O}(p_1,\ldots,p_{l+1})$ *then*

$$\deg_L \mathcal{F} = p_1 + \cdots + p_l + (l+1)p_{l+1}.$$

For $i = 0, \ldots, l$:
The i^{th} *degree of* $\mathcal{F}$ and the i^{th} *slope of* $\mathcal{F}$ are

$$\deg_{\overrightarrow{e_i}} \mathcal{F} := c_1(\Upsilon_i) \cdot L^{n-1} \quad \text{and} \quad \mu_{\overrightarrow{e_i}}(\mathcal{F}) := \frac{\deg_{\overrightarrow{e_i}} \mathcal{F}}{\text{rank}\, \mathcal{F}}$$

so that $\deg_L \mathcal{F} = \sum_{i=0}^{l} \deg_{\overrightarrow{e_i}} \mathcal{F}$ and $\mu_L(\mathcal{F}) = \sum_{i=0}^{l} \mu_{\overrightarrow{e_i}}(\mathcal{F})$.

The $L-$*normalisation of* $\mathcal{F}$ is

$$\mathcal{F}_{L-\text{norm}} := \mathcal{F}(-\vec{k}_{\mathcal{F}})$$

where $\vec{k}_{\mathcal{F}} \in \mathbb{Z}^{l+1}$ has components $k^i_{\mathcal{F}} := \left\lceil \frac{\mu_{\overrightarrow{e_i}}(\mathcal{F})}{\delta_L(\overrightarrow{e_i})} \right\rceil$. Indeed:

$$1 - r.\delta_L(\overrightarrow{e_i}) \leq \deg_{\overrightarrow{e_i}} \mathcal{F}_{L-\text{norm}} \leq 0.$$

The $L-$slope of a sheaf is essentially bounded below by its $L-$normalisation:

Lemma 1. *Let $\vec{k}_{\mathcal{F}}$ be the $L-$normalisation vector of $\mathcal{F}$, and set $\delta_L(L) := \deg_L(L)$; then*

$$\mu_L(\mathcal{F}) > \delta_L(\vec{k}_{\mathcal{F}}) - \delta_L(L) + t_{\mathcal{F}}$$

where $t_{\mathcal{F}} := \left\lceil \mu_L(\mathcal{F}) - \delta_L(\vec{k}) + \delta_L(L) - 1 \right\rceil \geq 0.$

Theorem 4 (Polycyclic Hoppe Criterion). *Let $\mathcal{E} \to X$ be a holomorphic vector bundle of rank $r \geq 2$ over a polycyclic variety with* Pic $(X) \simeq \mathbb{Z}^{l+1}$ *and polarisation L; define the constant $t_s := t_{\wedge^s \mathcal{E}}$ as by Lemma 1. If*

$$H^0(X, (\wedge^s \mathcal{E})_{L-\text{norm}}(\vec{p})) = 0 \qquad (*)$$

for all $\vec{p} \in \mathbb{Z}^{l+1}$ such that

$$\delta_L(\vec{p}) < \delta_L(L) - t_s \qquad (i)$$

then $\mathcal{E}$ is stable.
Conversely, if $\mathcal{E}$ is stable then

$$H^0(X, \mathcal{E}(\vec{p})) = 0, \ \forall \vec{p} \text{ such that } \delta_L(\vec{p}) \leq -\mu_L(\mathcal{E}).$$

Proof. Suppose $\mathcal{F} \hookrightarrow \mathcal{E}$ is a destabilising sheaf of rank s, such that $\det \mathcal{F} = \mathcal{O}_X(\vec{a})$. The inclusion induces a map $\wedge^s \mathcal{F} \hookrightarrow \wedge^s \mathcal{E}$ and so $H^0(X, (\wedge^s \mathcal{E})(-\vec{a})) \neq 0$, i.e.,

$$H^0(X, (\wedge^s \mathcal{E})_{L-\text{norm}}(\vec{k}_s - \vec{a})) \neq 0.$$

If, for $\vec{p} := \vec{k}_s - \vec{a}$, there could occur $\delta_L(\vec{p}) \geq \delta_L(L) - t_s$, then Lemma 1 would imply a contradiction:

$$\begin{aligned} \delta_L(\vec{a}) &= \deg \mathcal{F} \geq s\,\mu_L(\mathcal{E}) = \mu_L(\wedge^s \mathcal{E}) \\ &> \delta_L(\vec{k}_s) - \delta_L(L) + t_s \\ &\geq \delta_L(\vec{a}) \end{aligned}$$

using $\mu_L(\mathcal{E}) \leq \mu_L(\mathcal{F}) = \frac{\deg \mathcal{F}}{s}$; thus $\vec{p}$ satisfies (i).
Converse: trivial. □

Proposition 5 (Jardim-S.). *Let $\mathcal{E} \to \mathbb{P}^n \times \mathbb{P}^m, n \geq m > 1$ be a L-stable vector bundle obtained as the cohomology of a monad of the form*

$$0 \to (-1,0)^{\oplus a} \xrightarrow{\alpha} {}^{\oplus b} \oplus (-1,1)^{\oplus c} \xrightarrow{\beta} (0,1)^{\oplus a} \to 0 \qquad (2)$$

such that $c \leq a$ and $b + c - 2a = 2$. If $D \subset X$ is a polycyclic divisor of positive polydegree, then $\mathcal{E}|_D$ is $\mathcal{O}_D(1,1)$-stable.

¿ How about $\mathbb{P}^2 \times \mathbb{P}^1$!?

¿What if the vanishing hypothesis is too strong e.g.: monad cohomologies over $\mathbb{P}^2 \times \mathbb{P}^1$!?

A polycyclic variety X will be called a *polycyclic family over a cyclic variety* Z if it admits a projective morphism $X \xrightarrow{\pi} Z$ s.t. $\pi^*\mathrm{Pic}\,(Z) \hookrightarrow \mathrm{Pic}\,(X)$ is an injection.
So $\mathrm{Pic}\,(X) = \langle \Upsilon_0, \Upsilon_1, \ldots, \Upsilon_l \rangle \simeq \mathbb{Z}^{l+1}$ with $\Upsilon_0 \in |\pi^*(\mathcal{O}_Z(1))|$.
Given a bundle $Q \to X$, fix $z \in Z$ s.t. $Y_z := \pi^{-1}(z)$ has $\mathrm{Pic}\,(Y_z) = \langle \Upsilon_1, \ldots, \Upsilon_l \rangle$.

$$0 \longrightarrow Q(-d_0) \xrightarrow{\sigma_z} Q \xrightarrow{\rho_z} Q|_{Y_z} \longrightarrow 0$$

We will say that Q *has the restriction property at* z if sections restrict nontrivially to Y_z, i.e.,

$$0 \neq \sigma \in H^0(X, Q) \quad \Rightarrow \quad 0 \neq \rho_z(\sigma) \in H^0(Y_z, Q|_{Y_z}). \tag{3}$$

Definition 6. *Let $Q \to Y$ be a holomorphic bundle over a polycyclic variety with* Pic $(Y) \simeq \mathbb{Z}^l$*; then $\vec{v} \in \mathbb{Z}^l$ is a* bounding vector *for Q if, given $\vec{m} \in \mathbb{Z}^l$,*

$$m_i \leq -v_i \text{ for some } 1 \leq i \leq l \quad \Rightarrow \quad H^0(Y, Q(\vec{m})) = 0.$$

Corollary 7. *If, moreover, X is a polycyclic family over Z admitting a point $z \in Z$ such that, for each $1 \leq s \leq r-1$, the bundle $\wedge^s G$ admits a bounding vector $\overrightarrow{v_s} = \overrightarrow{v_s}(z)$ and has the restriction property (3) at z, then it suffices to check $(*)$ for all $\vec{p}$ satisfying both (i) and:*

$$p^i < k_s^i + v_s^i, \quad i = 1, \cdots, l \qquad (ii)$$

where $\vec{k}_s := \vec{k}_{\wedge^s G}$ is the $L-$normalisation vector.

Examples: extensions over $\mathbb{P}^2 \times \mathbb{P}^1$

Given integers $p \geq s$ and $t \geq q+2$, we obtain a bundle $\mathcal{E} \to \mathbb{P}^2 \times \mathbb{P}^1$ as a non-trivial extension of the form:

$$0 \longrightarrow \mathcal{O}(p,q) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(s,t) \longrightarrow 0\,.$$

A judicious choice of p, q, s, t guarantees that $\mathcal{E}$ is asymptotically stable [Jardim-Prata-S.], e.g.:

$$0 \longrightarrow \mathcal{O}(-1,-1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(-1,1) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}(-1,0) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(-1,2) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}(-1,1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}(-1,3) \longrightarrow 0$$

... and many more!

NB.: Required a *polycyclic* stability theory.

¿How about *rigid* asymptotically stable examples?

¿Can we extend the theory to 'asymptotically stable' reflexive sheaves?

Thank you!

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