Supersymmetric localization and the gauge/gravity duality

Dario Martelli

King's College London

European Research Council

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Gauge Theories in Higher Dimensions

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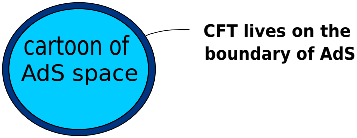
Outline

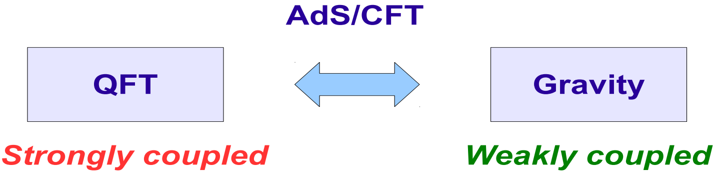
1. Introduction
2. Part I: supersymmetric localization
3. Part II: gauge/gravity duality

I will focus on four-dimensional field theories and five-dimensional gravity duals

Gauge/Gravity duality

Equivalence between (quantum) **gravity** in bulk space-times and **quantum field theories** on their boundaries





Localization

- For certain supersymmetric field theories defined on compact curved Riemannian manifolds the path integral may be computed exactly
- Localization: functional integral over all fields of a theory $\longrightarrow$ integral/sum over a reduced set of field configurations
- Saddle point around a supersymmetric locus gives the exact answer
- A priori the path integral ("partition function" **Z**) depends on the parameters of the theory and of the background geometry

Supersymmetry

- When bulk and boundary are supersymmetric we can perform detailed computations on both sides and (in certain limits) compare them

- Supersymmetry in the bulk ⇒ **supersymmetric solutions of supergravity equations**

- Supersymmetry on the boundary ⇒ **“rigid” curved space supersymmetry**

Part I: Supersymmetric localization

Prequel: localization on the round three-sphere

- Supersymmetric localization attributed to [Pestun]: $\mathcal{N} = 2$ four-dimensional QFT on round $\mathbf{S^4}$ $\rightarrow$ followed by results in three dimensions [J. Sparks' talk]

- Any $\mathbf{d = 3}$, $\mathcal{N} = 2$ gauge theory on the round $\mathbf{S^3}$, preserves supersymmetry [Kapustin-Willet-Yaakov], [Jafferis], [Hama-Hosomichi-Lee]. Key ingredient: on the (unit-radius) round $\mathbf{S^3}$ there exist Killing spinors χ

$$\nabla_i \chi = \frac{i}{2}\gamma_i \chi$$

- Full path integral $\rightarrow$ matrix integral with integrand a super-determinant where "most", but not all eigenvalues cancel out:

$$\frac{\det D_{\mathrm{ferm}}}{\det D_{\mathrm{bos}}} = \frac{\prod \text{ferm eigenvalues}}{\prod \text{bos eigenvalues}} = \frac{\prod \text{unpaired ferm eigenvalues}}{\prod \text{unpaired bos eigenvalues}}$$

Four dimensional $\mathcal{N} = 1$ supersymmetric field theories

- Main character: $\mathbf{d = 4}$ supersymmetric gauge theories with "matter"
- Supersymmetry organises the fields in multiplets, containing fields with different spin
- Vector multiplet: gauge field $\boldsymbol{\mathcal{A}}$ (connection on a bundle); Weyl spinor $\boldsymbol{\lambda}$; "auxiliary" scalar **D** (sort of Lagrange multiplier), all transforming in the adjoint representation of a group **G**
- Chiral multiplet (the "matter"): complex scalar ϕ; Weyl spinor ψ; "auxiliary" scalar **F**, all transforming in a representation $\boldsymbol{\mathcal{R}}$ of the group **G**
- In flat space with Lorentzian signature, supersymmetric Lagrangians containing these fields are text-book material (Euclidean space has some extra caveats)

Four dimensional $\mathcal{N} = 1$ supersymmetric field theories

- For example, defining $\mathbf{D}_\mu = \partial_\mu - \mathbf{i}\mathcal{A}_\mu \cdot$, where $\cdot$ denotes action on the appropriate representation, we have

$$\mathcal{L} = (\mathbf{D}^\mu \phi)^\dagger \mathbf{D}_\mu \phi + \mathbf{i}\psi^\dagger \sigma^\mu \mathbf{D}_\mu \psi + \ldots$$

- Somewhat strangely, rigid supersymmetry in curved space (Euclidean or Lorentzian) addressed systematically only in the 2010's
- But local supersymmetry studied since long time ago $\rightarrow$ supergravity
- [Festuccia-Seiberg]: take supergravity with some gauge and matter fields and appropriately throw away gravity $\rightarrow$ "rigid limit". Simple but correct
- Important: in the process of throwing away gravity, some extra fields of the supergravity multiplet remain, but are non-dynamical $\rightarrow$ background fields

Rigid supersymmetric four-manifolds

- Rigid limit of "new minimal" supergravity $\rightarrow$ Killing spinor equation for $\mathbf{d = 4}$, $\mathcal{N} = \mathbf{1}$ gauge theories on curved space

$$(\nabla_\mu - iA_\mu)\,\zeta + iV_\mu \zeta + iV^\nu \sigma_{\mu\nu}\zeta = 0$$

- The $\mathbf{A}_\mu, \mathbf{V}_\mu$ are background fields and ζ is a supersymmetry parameter

- In Euclidean signature: equivalent to Hermitian metric [Klare-Tomasiello-Zaffaroni], [Dumitrescu-Festuccia-Seiberg]

- In Lorentzian signature: equivalent to null conformal Killing vector [Cassani-Klare-DM-Tomasiello-Zaffaroni]

- Main motivation: localization in four dimensional $\mathcal{N} = \mathbf{1}$ gauge theories $\rightarrow$ [Assel-Cassani-DM]

Localization on four-manifolds: strategy outline

- Work in Euclidean signature and start with generic background fields $\mathbf{A}_\mu$, $\mathbf{V}_\mu$ associated to a Hermitian manifold
- Construct "susy-exact" Lagrangians for the vector and chiral multiplets $\rightarrow$ set-up localization on a general Hermitian manifold
- Restrict to backgrounds admitting a second spinor $\tilde{\zeta}$ with opposite R-charge $\rightarrow$ show that is possible to pick a real $\mathbf{A}$
- Further restrict to manifolds with topology $\mathbf{M_4} \simeq \mathbf{S^1} \times \mathbf{S^3}$
- Prove that the localization locus is given by gauge field $\mathcal{A}_\tau =$ constant, with all other fields $(\lambda, \mathbf{D}; \phi, \psi, \mathbf{F})$ vanishing
- Partition function reduces to a matrix integral over the Kaluza-Klein (Fourier) modes of $\mathcal{A}_\tau$ on $\mathbf{S^1}$ $\rightarrow$ integrand is infinite product of 3d super-determinants $\rightarrow$ use the 3d results! [J. Sparks' talk]

Localizing Lagrangians and saddle point equations

- The bosonic parts of the localizing terms constructed with ζ are

$$\mathcal{L}^{(+)}_{\text{vector}} = \text{tr}\left(\frac{1}{4}\mathcal{F}^{(+)}_{\mu\nu}\mathcal{F}^{(+)\,\mu\nu} - \frac{1}{4}\mathsf{D}^2\right)$$

$$\mathcal{L}_{\text{chiral}} = \left(\mathsf{g}^{\mu\nu} - \mathsf{i}\mathsf{J}^{\mu\nu}\right)\mathsf{D}_\mu\widetilde{\phi}\mathsf{D}_\nu\phi + \widetilde{\mathsf{F}}\mathsf{F}$$

Where $\mathsf{D}_\mu = \nabla_\mu - \mathsf{i}\mathsf{q}_\mathsf{R}\mathsf{A}_\mu - \mathsf{i}\mathcal{A}_\mu$.

- In Euclidean signatures all fields are doubled, and to evaluate the path integral one needs to impose reality conditions

- With the obvious ones, $\mathcal{A}, \mathsf{D}$ Hermitian, $\widetilde{\phi} = \phi^\dagger$, $\widetilde{\mathsf{F}} = \mathsf{F}^\dagger$, we obtain the saddle point equations

vector :	$\mathcal{F}^{(+)}_{\mu\nu} = 0\,,$	$\mathsf{D} = 0$
chiral :	$\mathsf{J}^\mu{}_\nu\mathsf{D}^\nu\widetilde{\phi} = \mathsf{i}\mathsf{D}^\mu\widetilde{\phi}\,,$	$\mathsf{F} = 0$

Aside: localization on general Hermitian manifolds

- The saddle-point condition of the vector multiplet is the instanton equation on $\mathbf{M_4}$. I don't have to explain this equation at this workshop!

$$\mathbf{Z} = \sum_{\text{charge n inst.}} \int_{\text{inst. moduli space}} [\text{inst. measure}]\ \mathbf{Z}_{\text{classic}}\ \mathbf{Z}_{1-\text{loop}}$$

- Instantons on Hermitian manifolds (HYM) $\rightarrow$ hard problem (?)

- The saddle-point condition of the chiral multiplet can be rewritten as $\bar{\partial}_{\mathbf{D}}\phi = 0 \rightarrow$ holomorphic sections of instanton bundle (+ further twist)

- Curiously, it is possible to deform the instanton equation to obtain the "vortex" equations [Bradlow], [Garciá-Prada]

$$\mathbf{J}_{\mu\nu}\mathcal{F}^{\mu\nu} = \phi^\dagger\phi + \tau\ , \qquad \Omega_{\mu\nu}\mathcal{F}^{\mu\nu} = 0$$
$$\bar{\partial}_{\mathbf{D}}\phi = 0$$

- Exploited in physics to perform an alternative localization in some cases ("Higgs branch" localization), [Benini-Cremonesi,...]

Geometries with two supercharges of opposite R-charge

- Assume that there exist a second spinor $\widetilde{\zeta}$, with opposite chirality, obeying the rigid new minimal equation

$$(\nabla_\mu + iA_\mu)\,\widetilde{\zeta} - iV_\mu\widetilde{\zeta} - iV^\nu\widetilde{\sigma}_{\mu\nu}\widetilde{\zeta} = 0$$

- Geometry is a special case of ambihermitian manifold, which may be neatly characterised by the complex holomorphic Killing vector field $K^\mu = \zeta\sigma^\mu\widetilde{\zeta}$

- The metric takes a canonical form in terms of complex coordinates z, w

$$ds^2 \;=\; \Omega^2[(dw + hdz)(d\bar{w} + \bar{h}d\bar{z}) + c^2dzd\bar{z}]$$

with $\Omega(z,\bar{z})$, $c(z,\bar{z})$, $h(z,\bar{z})$ arbitrary functions

Choice of real $\mathbf{A}$

- The background fields take the form

$$\mathbf{V} = \mathrm{d}^c \log \Omega + \frac{2}{\Omega^2 \mathbf{c}^2} \operatorname{Im}\left(\partial_{\bar{z}} \mathbf{h}\, \mathbf{K}\right) + \kappa \mathbf{K}$$

$$\mathbf{A} = \frac{1}{2} \mathrm{d}^c \log\left(\Omega^3 \mathbf{c}\right) + \frac{1}{2} \mathrm{d}\omega + \left(\frac{3}{2}\kappa - \frac{\mathrm{i}}{\Omega^2 \mathbf{c}^2} \partial_{\bar{z}} \mathbf{h}\right) \mathbf{K}$$

- ω is a phase entering in the Killing spinors, that can be fixed requiring **A** to be globally well-defined
- κ is an arbitrary function a priori, that drops out from the rigid supersymmetry equations $\longrightarrow$ refer to as "κ-gauge"
- We fix κ so that the last term in **A** vanishes and therefore **A** is real

"Toric" Hopf surfaces

- A Hopf surface is essentially a four-dimensional complex manifold with the topology of $\mathbf{S^1} \times \mathbf{S^3}$, and it may be defined as a compact complex surface whose universal covering is $\mathbb{C}^2 - (0, 0)$

- Described as quotient of $\mathbb{C}^2 - (0, 0)$, with coordinates z_1, z_2 identified as

$$(z_1, z_2) \sim (pz_1, qz_2)$$

  where p, q are in general two complex parameters

- We show that on a Hopf surface we can take a very general metric

$$ds^2 = \Omega^2 d\tau^2 + f^2 d\rho^2 + m_{IJ} d\varphi_I d\varphi_J \qquad I, J = 1, 2$$

  while preserving two spinors ζ and $\widetilde{\zeta}$

- τ is a coordinate on $\mathbf{S^1}$, while the 3d part has coordinates $\rho, \varphi_1, \varphi_2$, describing $\mathbf{S^3}$ as a $\mathbf{T^2}$ fibration over an interval $\rightarrow$ "toric"

The matrix model

- The localizing locus simplifies drastically, due to "doubling the equations imposed", e,g. $\rightarrow \mathcal{F}^{+} = \mathcal{F}^{-} = 0 \rightarrow$ full contribution comes from zero-instanton sector! Flat connections $\mathcal{A}_{\tau} =$ constant, and all other fields vanishing

- The localized path integral is reduced to exactly the same 3d computation done in [Alday-DM-Richmond-Sparks] (with no CS terms). More precisely, to an infinite product of that, one for each KK supermultiplet mode

- The Hopf surface complex structure data $\mathbf{p}, \mathbf{q}$ maps to the almost contact structure data $\mathbf{b_1}, \mathbf{b_2}$ as: $\mathbf{p} = \mathrm{e}^{-2\pi|\mathbf{b_1}|}$, $\mathbf{q} = \mathrm{e}^{-2\pi|\mathbf{b_2}|}$

- Infinite products regularised using fancy mathematical formulas. E.g.

$$Z^{\text{chiral}}_{\text{1-loop}} = \prod_{\rho\in\Delta_{\mathcal{R}}} \prod_{n\in\mathbb{Z}} Z^{\text{chiral}}_{\text{1-loop (3d)}}\left[\sigma_0^{(n,\rho)}\right]$$

$$\rightarrow \ \mathrm{e}^{\mathrm{i}\pi\Psi^{(0)}_{\text{chi}}}\,\mathrm{e}^{\mathrm{i}\pi\Psi^{(1)}_{\text{chi}}} \prod_{\rho\in\Delta_{\mathcal{R}}} \Gamma_{\mathrm{e}}\left(\mathrm{e}^{2\pi\mathrm{i}\rho_{\mathcal{A}_0}}\,(\mathbf{pq})^{\frac{r}{2}}, \mathbf{p}, \mathbf{q}\right)$$

Supersymmetric index

- The prefactor $\Psi^{(1)}_{\mathrm{chi}}$ is anomalous and must cancel after combining with the vector multiplet contribution $\rightarrow$ anomaly cancellation conditions "for free"

- The rest combines into the following formula

$$\mathrm{Z}[\mathcal{H}_{\mathrm{p,q}}] \; = \; \mathrm{e}^{-\mathcal{F}(\mathrm{p,q})} \, \mathcal{I}(\mathrm{p,q})$$

where $\mathcal{I}(\mathrm{p,q})$ is the supersymmetric index with $\mathbf{p},\mathbf{q}$ fugacities

$$\mathcal{I}(\mathrm{p,q}) \; = \; \frac{(\mathrm{p;p})^{\mathrm{r_G}}(\mathrm{q;q})^{\mathrm{r_G}}}{|\mathcal{W}|} \int_{\mathrm{T^{r_G}}} \frac{\mathrm{dz}}{2\pi \mathrm{iz}} \prod_{\alpha\in\Delta_+} \theta\left(\mathrm{z}^{\alpha},\mathrm{p}\right) \theta\left(\mathrm{z}^{-\alpha},\mathrm{q}\right) \prod_{\mathrm{J}} \prod_{\rho\in\Delta_{\mathrm{J}}} \Gamma_{\mathrm{e}}\left(\mathrm{z}^{\rho}(\mathrm{pq})^{\frac{\mathrm{r_J}}{2}},\mathrm{p,q}\right)$$

which may be defined as a sum over states as

$$\mathcal{I}(\mathrm{p,q}) \; = \; \mathrm{Tr}[(-1)^{\mathrm{F}} \mathrm{p}^{\mathrm{J}+\mathrm{J}'-\frac{\mathrm{R}}{2}} \mathrm{q}^{\mathrm{J}-\mathrm{J}'-\frac{\mathrm{R}}{2}}]$$

- The fact that the index is computed by the localized path integral on a Hopf surface was anticipated by [Closset-Dumitrescu-Festuccia-Komargodski]

Supersymmetric Casimir energy

- The path integral + regularisation produced an extra pre-factor $\mathcal{F}(\mathbf{p},\mathbf{q})$ explicitly given by

$$\mathcal{F}(\mathbf{p},\mathbf{q}) = \frac{4\pi}{3}\left(|\mathbf{b}_1| + |\mathbf{b}_2| - \frac{|\mathbf{b}_1| + |\mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}\right)(\mathbf{a}-\mathbf{c}) + \frac{4\pi}{27}\frac{(|\mathbf{b}_1| + |\mathbf{b}_2|)^3}{|\mathbf{b}_1||\mathbf{b}_2|}(3\,\mathbf{c} - 2\,\mathbf{a})$$

where

$$\mathbf{a} = \frac{3}{32}\left(3\,\mathrm{tr}\mathbf{R}^3 - \mathrm{tr}\mathbf{R}\right)\,, \qquad \mathbf{c} = \frac{1}{32}\left(9\,\mathrm{tr}\mathbf{R}^3 - 5\,\mathrm{tr}\mathbf{R}\right)$$

- Invariant depending only on complex structure and the trace anomaly coefficients $\mathbf{a},\mathbf{c}$ $\to$ should not be merely a "counterterm", expect to encode physical/mathematical properties

- We argued that it is essentially the "vacuum energy" $\to$ refer to as supersymmetric Casimir energy $\mathbf{E}_{\mathrm{susy}}$

More comments on the supersymmetric Casimir energy

- How does one know the result does not depend on the regularisation procedure, e.g. zeta-function?
- One must show that there are no finite, supersymmetric, "counterterms" – integrals of local densities
- Conjecture: there are no finite local counterterms (some exist, but vanish) [Assel-Cassani-DM] (unpublished)
- Supersymmetric Casimir energy can be recovered from the Hamiltonian formalism [Lorenzen-DM] (to appear)

$$\langle 0|\mathbf{H_{BPS}}|0\rangle = \mathbf{E}_{\mathrm{susy}}$$

where $\mathbf{H_{BPS}}$ is an appropriate supersymmetric Hamiltonian, such that $[\mathbf{H_{BPS}}, \mathbf{Q}_{\mathrm{susy}}] = 0$

Part II: Gauge/gravity duality

Constructing gravity duals

Idea: find a supersymmetric filling $\mathbf{M_5}$ of a given $\mathbf{M_4}$ in the context of $\mathbf{d=5}$, gauged supergravity, and use the fact that any such solution uplifts to a supersymmetric solution $\mathbf{M_5 \times Y_5}$ of Type IIB supergravity

Action*: $$\mathbf{S} = \frac{1}{16\pi G}\int \left[d^5x\sqrt{g}\left(R - F^2 + \frac{12}{\ell^2}\right) - \frac{8}{3\sqrt{3}} A \wedge F \wedge F \right]$$

KSE: $$\left[\nabla_\mu + \frac{i}{4\sqrt{3}}\left(\gamma_\mu{}^{\nu\lambda} - 4\delta_\mu^\nu \gamma^\lambda\right) F_{\nu\lambda} - \frac{1}{2\ell}\left(\gamma_\mu - 2\sqrt{3}\, i A_\mu\right)\right]\epsilon = 0$$

Dirichlet problem: find $(\mathbf{M_5}, g_{\mu\nu}, \mathbf{A})$ such that

- The conformal boundary of $\mathbf{M_5}$ is $\mathbf{M_4}$
- The gauge field $\mathbf{A}$ restricts to $\mathbf{A}^{cs} = \mathbf{A}^{(4)} - \frac{3}{2}\mathbf{V}^{(4)}$
- The Killing spinor ϵ restricts to the Killing spinor χ

Check: The on-shell sugra action should reproduce the Casimir energy!

*From now on, $\mathbf{A}$ will denote the five-dimensional gravi-photon field, while the four-dimensional background fields $\mathbf{A}^{(4)}$, $\mathbf{V}^{(4)}$ will not appear in the formulas

3d/4d gravity duals

- This can be repeated with "4d" replaced by "3d" and "5d" replaced by "4d" almost step by step: in fact, this is where we started from [J. Sparks' talk]
- Solutions constructed by: [DM,Passias,Sparks,Farquet,Lorenzen] and some variations by [Huang-Rey-Zhou;Nishioka]
- In $\mathbf{d = 3}$ field theories on $\mathbf{M_3 \simeq S^3}$, the large $\mathbf{N}$ limit of the localized partition function matches exactly the $\mathbf{d = 4}$ supergravity action, evaluated on a solution $\longrightarrow$ perfect cross-check of gauge/gravity and localization!
- This "sets the standard" for similar constructions in different dimensions

5d gravity duals

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- Is there any known example of a 5d gravity solution whose conformal boundary is a Hermitian manifold? Yes: Euclidean global AdS_5, with conformal boundary the round $\mathbf{S^1} \times \mathbf{S^3}$
- Gravity dual of a generic Hermitian manifold is a very hard problem, e.g. no isometries. Moreover, no localization results (yet) so there is nothing to compare with
- Start investigating solutions whose conformal boundary $\mathbf{M_4}$ is a more general Hopf surface, thus $\mathbf{M_4} \simeq \mathbf{S^1} \times \mathbf{S^3}$
- Useful technical simplification: $\mathbf{SU(2)} \times \mathbf{U(1)} \times \mathbf{U(1)}$ symmetry $\rightarrow$ ODE's $\rightarrow$ singles out $\mathbf{S^1} \times \mathbf{S^3_{squashed}}$
- We looked for a supersymmetric "filling" $\mathbf{M_5}$ of this boundary, in minimal gauged supergravity in $\mathbf{d = 5}$ [Cassani-DM]

Gutowski-Reall equation

Existence of one solution ϵ yields a canonical form of the metric and the gauge field [Gauntlett-Gutowski]. In the "time-like" class the metric reads

$$\mathrm{d}s^2 = -f^2(\mathrm{d}y + \omega)^2 + f^{-1}\mathrm{d}s_B^2$$

where $\mathrm{d}s_B^2$ is a Kähler metric and $\frac{\partial}{\partial y}$ is a time-like (in the bulk) Killing vector
Further imposing an ansatz with $\mathbf{SU(2)} \times \mathbf{U(1)} \times \mathbf{U(1)_y}$ symmetry, with metric

$$\mathrm{d}s_B^2 \;=\; \mathrm{d}\rho^2 + a^2(\hat{\sigma}_1^2 + \hat{\sigma}_2^2) + (2aa')^2\hat{\sigma}_3^2$$

relates all functions in the ansatz, e.g.

$$f^{-1} = \frac{\ell^2}{12a^2a'}[4(a')^3 + 7a\,a'a'' - a' + a^2a''']$$

reducing the susy conditions to one ODE for one function $\mathbf{a(\rho)}$. This is the ODE derived by [Gutowski-Reall], who also found a one-parameter family of black-hole solutions, i.e. with event horizon

We found a new one-parameter solution s.t. [CM] $\cap$ [GR] $=$ AdS_5

The solution

Solve the ODE order by order by plugging a "UV" expansion ($\rho \to \infty$)

$$\mathbf{a} = \mathbf{a_0 e^{\rho}} \left[\mathbf{1 + (a_2 + c\rho) \frac{e^{-2\rho}}{a_0^2} + \left(a_4 + a_{4,1}\rho + a_{4,2}\rho^2\right) \frac{e^{-4\rho}}{a_0^4} + \dots} \right]$$

and an "IR" expansion ($\rho \to \mathbf{0}$)

$$\mathbf{a = a_0^{IR} + a_1^{IR}\rho + a_2^{IR}\rho^2 + a_3^{IR}\rho^3 + \dots}$$

Require that in the UV the solution is AlAdS$_5$ and in the IR it is smooth with no horizon. Globally $\mathbb{R}^{1,4} = \mathbb{R} \times \mathbb{R}^4$

UV: five free parameters $\mathbf{a_0, c, a_2, a_4, a_6}$. IR: one free parameter ξ. A solution interpolating between IR and UV is shown to exist

1. analytically as a linearised (in ξ) perturbation of AdS$_5$
2. numerically for arbitrary values of the deformation parameter ξ

Linearised solution

We obtain

$$f = 1 + \frac{2\xi \log \cosh \rho}{\sinh^2 \rho} + \mathcal{O}(\xi^2)$$

After the change of coordinate $\hat{\psi} = \psi - \frac{2}{1-4c}\,t$, $y = t$ the metric reads

$$ds^2 = d\rho^2 - \cosh^2\rho dt^2 + \frac{1}{4}\sinh^2\rho\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) + [\text{explicit } \mathcal{O}(\xi)]$$

and the gauge field

$$A = \frac{1}{2\sqrt{3}}dt - \frac{\sqrt{3}}{4}\xi\left[1 - \frac{2\log\cosh\rho}{\sinh^2\rho}\right]\sigma_3 + \mathcal{O}(\xi^2)$$

All the UV parameters are expressed in terms of the single IR parameter:

$$a_0 = \frac{1}{4} + \frac{\xi}{16}(1 - 4\log 2) + \mathcal{O}(\xi^2) \qquad a_2 = -\frac{1}{16} - \frac{3\,\xi}{32}(1 + 4\log 2) + \mathcal{O}(\xi^2)$$

$$a_4 = \frac{3\,\xi}{32}\left(\frac{3}{16} - \log 2\right) + \mathcal{O}(\xi^2) \qquad a_6 = \frac{\xi}{512}\left(\frac{113}{48} - 7\log 2\right) + \mathcal{O}(\xi^2)$$

$$c = \frac{3}{8}\,\xi + \mathcal{O}(\xi^2)$$

Numerics

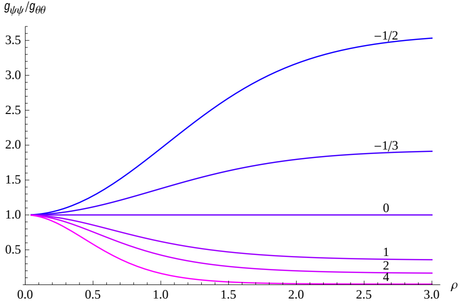

Figure: Different values of the IR parameter ξ are indicated on the curves. Asymptotically, this shows the value of the parameter $\mathbf{v^2 = 1 - 4c}$, controlling the squashing of the boundary $\mathbf{S^3_v}$

$$\mathbf{ds^2_{bdry} = (2a_0)^2 \left[-\frac{1}{v^2} dt^2 + \frac{1}{4} \left(\sigma_1^2 + \sigma_2^2 + v^2 \sigma_3^2 \right) \right]}$$

Relation between **UV** and **IR** parameters

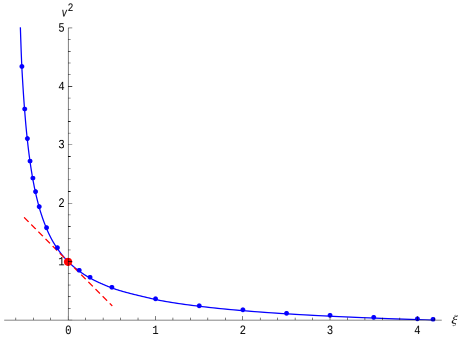

Figure: The squashing $\mathbf{v}$ ranges between 0 and ∞ for $\mathbf{4.2} \gtrsim \boldsymbol{\xi} \gtrsim \mathbf{-0.7}$. The red line represents the relation obtained from the linearised analysis around the AdS_5 solution at $\boldsymbol{\xi} = \mathbf{0}$

The renormalised on-shell action

- The holographically renormalised on-shell action is

$$S_{\rm ren} = \lim_{\rho\to\infty} \left(S_{\rm bulk} + S_{\rm GH} + S_{\rm ct}\right)$$

- The on-shell bulk action can be written as

$$S_{\rm bulk} = -\frac{1}{2\pi G\ell^2}\int d^5x\sqrt{g} - \frac{1}{12\pi G}\int d(A\wedge *_5F)$$

  where the second term is a total derivative. In fact, also the first one is, so that the bulk on-shell action reduces to a boundary term

- Notice $S_{\rm bulk}$ depends on the gauge for A. Under a gauge transformation $\delta A = \delta A_t\, dt$, where δA_t is a constant, the on-shell action changes by

$$\delta S_{\rm bulk} = -\frac{\delta A_t}{12\pi G}\int dt \int_{S^3_{\rm bdry}} *_5F$$

- In some previous formulas for A we picked a specific gauge for A

Euclidean on-shell action

- This gauge is such that $\mathcal{L}_{\frac{\partial}{\partial t}}\zeta = \mathcal{L}_{\frac{\partial}{\partial t}}\epsilon = 0$, therefore the spinors are t independent. In any other gauge, the spinor acquires a phase $\sim e^{i\delta A_t t}$
- In this gauge, we can do a simple analytic continuation $t \to it$, to obtain a boundary Euclidean geometry with $S^1 \times S^3$ topology
- The Euclidean boundary metric and gauge field are

$$ds^2_{\rm bdry} = \frac{1}{v^2}dt^2 + \frac{1}{4}\left(\sigma_1^2 + \sigma_2^2 + v^2\sigma_3^2\right)$$

$$\frac{\sqrt{3}}{\ell}A_{\rm bdry} = \frac{i}{2\ell}dt + \frac{1}{2}(v^2 - 1)\,\sigma_3$$

- Both the bulk metric and the bulk gauge field become complex. The analytically continued on-shell action remains real and reads

$$I = \frac{\pi\ell^2\Delta_t}{G}\left[\frac{2}{27v^2} + \frac{2}{27} - \frac{13}{108}v^2 + \frac{19}{288}v^4\right]$$

Holographic energy-momentum and R-current

- Computing the holographic energy-momentum tensor

$$\mathbf{T_{ij}} = -\frac{2}{\sqrt{\mathbf{h}}}\frac{\delta \mathbf{S}_{\mathrm{reg}}}{\delta \mathbf{h}^{\mathbf{ij}}}$$

the holographic trace anomaly vanishes $\langle \mathbf{T}^{\mathbf{i}}_{\mathbf{i}} \rangle = \mathbf{0}$, and there is no log divergence in $\mathbf{I}$, in agreement with [Cassani-DM]

- From the holographic R-symmetry current

$$\mathbf{j}^{\mathbf{i}} = \frac{1}{\sqrt{\mathbf{h}}}\frac{\delta \mathbf{S}_{\mathrm{reg}}}{\delta \mathbf{A}_{\mathbf{i}}}$$

we can compute the associated holographic conserved charge (R-charge)

$$\mathbf{Q} = \int_{\Sigma} \mathrm{d}^3\mathbf{x}\sqrt{\gamma}\,\mathbf{u_i}\langle \mathbf{j}^{\mathbf{i}} \rangle \;=\; \frac{1}{4\pi \mathbf{G}}\int_{\Sigma}\left(*_5\mathbf{F} + \frac{4}{3\sqrt{3}}\mathbf{A}\wedge\mathbf{F}\right)$$

Partition function and supersymmetric index

- Master formula of the AdS/CFT correspondence:

$$e^{-S_{\rm gravity}[M_5]} = Z_{\rm QFT}[M_4 = \partial M_5] \qquad \text{for} \qquad N \to \infty$$

- $M_5 \simeq S^1 \times \mathbb{R}^4 \Rightarrow$ path integral on $M_4 \simeq S^1 \times S^3$ with periodic boundary conditions for the fermions on S^1, is precisely the supersymmetric partition function on $M_4 = \mathcal{H}_{p,q} \simeq S^3$

$$Z[\mathcal{H}_{p,q}] \;=\; e^{-\mathcal{F}(p,q)}\,\mathcal{I}(p,q)$$

where $\mathcal{I}(p,q)$ is the supersymmetric index with p, q fugacities

- The (supersymmetric) Casimir energy may be defined as ($\beta \sim \log p \sim \log q$)

$$E_{\rm susy} \equiv -\lim_{\beta\to\infty} \frac{d}{d\beta} \log Z_{\rm QFT}[S^1_\beta \times M_3]$$

- Using known facts that $S_{\rm gravity} = \mathcal{O}(N^2)$ and in large N limit $\mathcal{I} = \mathcal{O}(N^0)$, we see the entire contribution comes from $\mathcal{F}(p,q)$!

Supersymmetric Casimir energy

- Our $\mathbf{M_4}$ turns out to be a Hopf surface with parameters

$$\mathbf{p} = \mathbf{q} = \mathrm{e}^{-\frac{\Delta_t}{\ell v^2}} \equiv \mathrm{e}^{-\beta}$$

so although the metric is a non-trivial deformation of the round case, the complex structure is essentially the standard one

- Inserting these values in our general formula for the pre-factor we obtain

$$-\mathcal{F} = \frac{4}{27}\beta\,(a + 3\,c)$$

- For a superconformal quiver with gravity dual ($\mathbf{a} = \mathbf{c} = \mathcal{O}(\mathbf{N}^2)$) we can compare $\mathbf{E}_{\mathrm{susy}}$ with the gravity side using standard formula relating the coefficient $\mathbf{a} = \mathbf{c}$ to the 5d Newton constant

$$\mathbf{E}_{\mathrm{susy}} = \frac{16}{27}\,a = \frac{2}{27}\frac{\pi\ell^3}{G} \quad \text{for} \qquad N \to \infty$$

Comparison with gravity

Fact: we have computed the holographically renormalised on-shell action (in the gauge where $\mathcal{L}_{\frac{\partial}{\partial t}}\epsilon = 0$), as a function of the squashing parameter

$$I = \frac{\Delta_t}{\ell v^2}\frac{\pi\ell^3}{G}\left[\frac{2}{27} + \frac{2}{27}v^2 - \frac{13}{108}v^4 + \frac{19}{288}v^6\right]$$

Fact: in the limit $v^2 = 1$ this reduces to $\frac{I}{\Delta_t} = \frac{3}{32}\frac{\pi\ell^2}{G}$, which [Balasubramanian-Kraus] interpreted as the "Casimir energy on S^3"

We would like to interpret the first red term in I as the relevant Casimir energy, to be compared with the field theory result

The $v^2 = 1$ limit of this gives $\frac{2}{27}\frac{\pi\ell^2}{G}$. However, the Casimir energy to which [Balasubramanian-Kraus] refer is $\langle 0|H|0\rangle$ of a Hamltonian different form H_{BPS}!

Presumably there are new finite holographic counterterms that must be included to render the full bulk+boundary sugra action supersymmetric. The following is just an example of how it might work...

Finite counterterms and ambiguities

- In $\mathbf{d = 5}$ holographic renormalisation does not determine unambiguously all the counterterms necessary to render the on-shell sugra action finite

- There are four independent types of standard counterterms, which are finite on removing the UV cut-off (because scale-invariant)

$$\Delta S = \frac{\ell^3}{8\pi G}\int_{\partial M} d^4x\sqrt{h}\left(\alpha\,\mathcal{E} + \beta\, C_{ijkl}C^{ijkl} + \gamma\, R^2 - \frac{\delta}{\ell^2}F_{ij}F^{ij}\right)$$
$$\propto \frac{\gamma}{4}\left(4 - v^2\right)^2 + \frac{1}{6}\left(8\beta - \delta\right)\left(1 - v^2\right)^2$$

- Has the correct polynomial dependence on $\mathbf{v^2}$ to remove the unwanted terms in $\mathbf{I}$. But there isn't a choice of γ, β, δ removing all terms simultaneously

- An independent term can be constructed using the complex structure: with the Ricci form of boundary geometry $\mathcal{R}_{ij} = \frac{1}{2}R_{ijkl}\mathcal{J}^{kl}$ we obtain

$$I - \frac{1}{108}\frac{\ell^3}{8\pi G}\int_{\partial M} d^4x\sqrt{h}\left(\frac{7}{24}R^2 + \frac{17}{\ell^2}F_{ij}F^{ij} - \mathcal{R}_{ij}\mathcal{R}^{ij}\right) = \frac{2}{27}\frac{\Delta_t}{\ell v^2}\frac{\pi\ell^3}{G}$$

Outlook

- Computed partition function on general four-dimensional Hopf surfaces $\rightarrow$ supersymmetric index + Casimir energy
- Five-dimensional gravity duals harder to construct explicitly then four-dimensional ones – we have obtained one non-trivial example
- Explore the role of the supersymmetric Casimir energy both in the field theory and in the gauge/gravity duality $\rightarrow$ reconcile with holographic renormalization
- Challenge: compute partition function of $\mathcal{N} = \mathbf{1}$ field theories on compact Hermitian manifolds (e.g. Kähler) $\rightarrow$ instantons, vortices, holomorphic invariants...