E-string elliptic genus from domain walls

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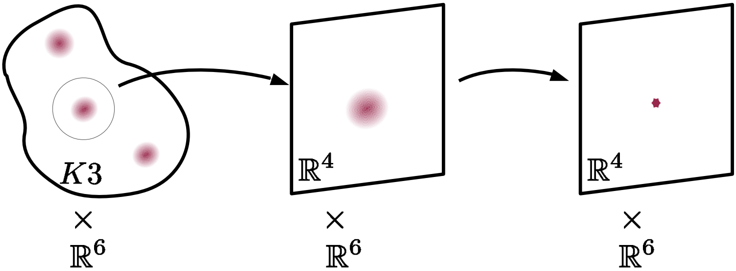
- 1406.0850 Haghighat, GL, Vafa
- 1305.6322 Haghighat, Iqbal, Kozçaz, GL, Vafa
- 1310.1185 Haghighat, Kozçaz, GL, Vafa

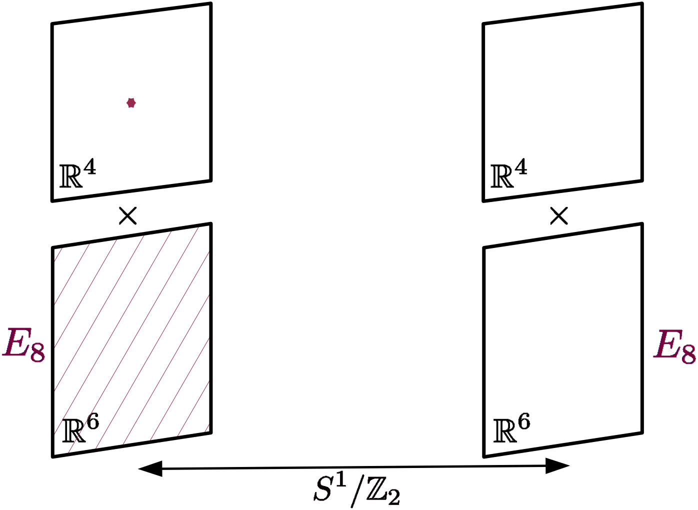
E-strings

E(xceptional)-string theory: 6d $\mathcal{N}=(1,0)$ SCFT living at the core of a small E_8 instanton of the $E_8\times E_8$ heterotic string [Witten, Ganor-Hanany,...].



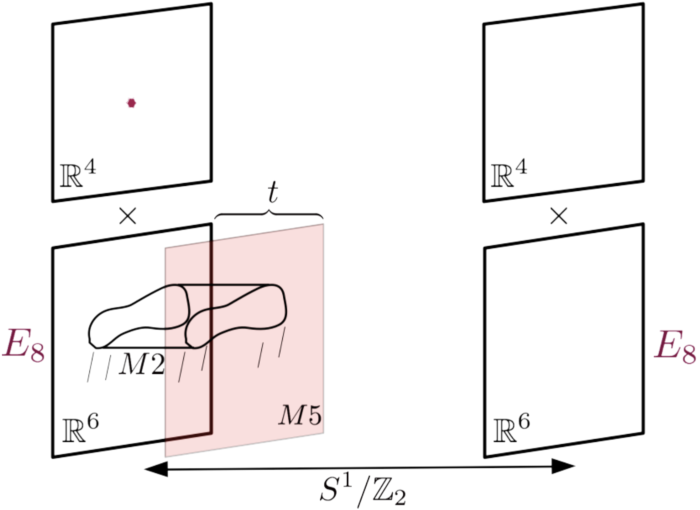
D.o.f.: tensionless strings with $\mathcal{N}=$ (2,0) w.s. susy, coupled to self-dual $B^+_{\mu\nu}$ ($\subset$ tensor multiplet T) $+$ $\widehat{E_8}$ current algebra.

Lift to M-theory on $\mathbb{R}^{10} \times S^1/\mathbb{Z}^2$:



'End of the world' M9 planes support E_8 gauge bundles.
E-string CFT lives on $\mathbb{R}^6 \subset M9_L$.

Lift to M-theory on $\mathbb{R}^{10} \times S^1/\mathbb{Z}^2$:



M2 branes $\sim$ E-strings. String tension $= t =$ vev of $\phi \subset T$ parametrizes 'Coulomb' branch.

Compactify the 6d theory on $S^1 \longrightarrow$ effective 5d $Sp(1) \sim SU(2)$ gauge theory with 8 fundamental hypermultiplets [Seiberg].

SO(16) flavor symmetry $\rightsquigarrow \widehat{E_8}$ at conformal fixed point.

States of E-strings wrapping $S^1 \leftrightarrow$ 5d electrically charged BPS states.

Next, put the 5d theory on a thermal circle, count the BPS states moving along it. It turns out:

$$Z_{5d}^{BPS} = \sum_{n=0}^{\infty} e^{-nt} Z_n^E.$$

Z_n^E = elliptic genus of n E-strings wrapping T^2:

$$Z_n = \mathrm{Tr}_R (-1)^F Q_\tau^{H_R} \overline{Q_\tau}^{H_L} \prod x_a^{K_a}$$

K_a = Cartan generators for spacetime twists + flavor symmetries.
$\tau = \frac{1}{2\pi i} \log Q_\tau$ = complex structure of the T^2 .

(supersymmetry on left-moving sector $\rightarrow Z_n$ is a holomorphic function of τ)

E-strings form bound states. Worldsheet theory for $n > 1$ is unknown. Only Z_1 is known exactly.

Aim of the talk: relate computation of E-string elliptic genus to that of other theories that arise from M2 branes:

- 10d $E_8 \times E_8$ heterotic (H) strings = M2 branes bounded by M9 planes.
- Self-dual strings of the 6d A_1 $\mathcal{N} = (2,0)$ theory (M-strings) = M2 branes bounded by M5 branes.

The plan:

- Argue that these elliptic genera can be computed by combining appropriate M5 or M9 domain wall factors.
- Review known results for H, M and E strings.
- Use domain wall picture to obtain an *exact* expression for elliptic genus of 2 E-strings.

M2 worldvolume $= T^2 \times \mathbb{R}$.

Let $\mathrm{Vol}(T^2) \to 0$. This leads to QM along $\mathbb{R}$. The Hamiltonian counts the number of M2 branes.

Can think of M5, M9 as domain walls for M2 branes on $T^2 \times \mathbb{R}$, since they fill T^2. For a collection of n M2's, M5 brane b.c. give QM states labeled by Young diagrams ν of size $|\nu| = n$ [Gomis-Rodriguez-Gomez-Van Raamsdonk-Verlinde, H.-C. Kim-S.Kim].

M5 branes give rise to operators:

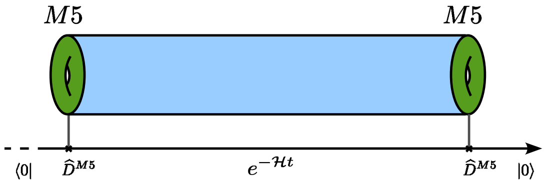
$$\text{M5 branes} \quad \rightsquigarrow \quad \widehat{D}^{M5}|\mu\rangle = \sum_{\nu} D^{M5}_{\nu\mu}|\nu\rangle.$$

M9 planes give rise to boundary states:

$$\text{M9 planes} \quad \rightsquigarrow \quad |M9\rangle = \sum_{\nu} D^{M9}_{\nu}|\nu\rangle + \dots$$

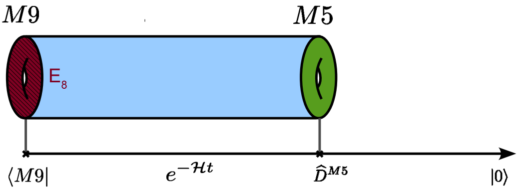
Little known about M9 b.c. (see [Berman-Perry-Sezgin-Thompson]).

Elliptic genus of strings is mapped to expectation values in the QM. For M-strings [1305.6322]:



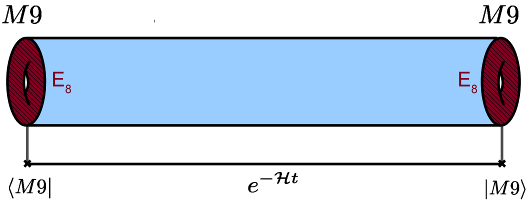
$$\sum_n e^{-nt} Z_n^M = \langle 0|\widehat{D}^{M5} e^{-\mathcal{H}t} \widehat{D}^{M5}|0\rangle \implies Z_n^M = \sum_{|\nu|=n} D^{M5}_{\emptyset\nu} D^{M5}_{\nu\emptyset}$$

E-string elliptic genus



$$\sum_n e^{-nt} Z_n^E = \langle M9 | e^{-\mathcal{H}t} \widehat{D}^{M5} | 0 \rangle \implies Z_n^E \sim \sum_{|\nu|=n} D_\nu^{M9,L} D_{\nu\emptyset}^{M5}$$

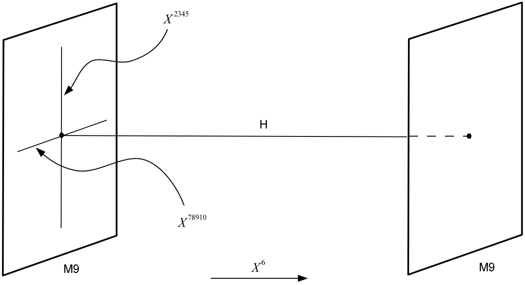
Heterotic string elliptic genus



$$\sum_n e^{-nt} Z_n^H = \langle M9 | e^{-\mathcal{H}t} | M9 \rangle \implies Z_n^H \sim \sum_{\alpha \in H} D_\alpha^{M9,L} D_\alpha^{M9,R}$$

Heterotic strings

$$T^2_{0,1} \times \mathbb{C}^2_{2,3,4,5} \times S^1_6/\mathbb{Z}_2 \times \mathbb{C}^2_{7,8,9,10}$$



Twisted sectors localized at $X_6 = 0, \pi \to E_{8,L} \times E_{8,R}$ current algebra on heterotic string worldsheet.

Supercharges:

$\Gamma^6\epsilon = \epsilon$ (from M9 planes), $\Gamma^{016}\epsilon = \epsilon$ (from M2 branes) $\implies$ $(8,0)$ w.s. SUSY.

Further broken by twisting as we go around cycles of T^2:

$$(z_1, z_2, z_3, z_4) \in \mathbb{C}^2_{2,3,4,5} \times \mathbb{C}^2_{7,8,9,10} \to (e^{2\pi i\epsilon_1} z_1, e^{2\pi i\epsilon_2} z_2, e^{2\pi i\epsilon_3} z_3, e^{2\pi i\epsilon_4} z_4)$$

(2,0) SUSY survives, provided $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0$.

Can also turn on twists $(\vec{m}_{E_8,L}, \vec{m}_{E_8,R})$ for the $E_8 \times E_8$ symmetry. Thus

$$Z^H_n = Z^H_n(\vec{m}_{E_8,L}, \vec{m}_{E_8,R}, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \tau).$$

One heterotic string:

$$Z_1^H = -\underbrace{\frac{\Theta_{E_8}(\tau;\vec{m}_{E_8,L})\Theta_{E_8}(\tau;\vec{m}_{E_8,R})}{\eta^{16}}}_{\text{16 bosons compactified on }\Gamma_{E_8\times E_8}} \cdot \underbrace{\frac{\eta^4}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)}}_{\text{8 spacetime bosons}},$$

where

$$\Theta_{E_8}(\tau;\vec{m}_{E_8}) = \sum_{\vec{k}\in\Gamma_{E_8}} \exp(\pi i\tau\vec{k}\cdot\vec{k} + 2\pi i\vec{m}_{E_8}\cdot\vec{k}) = \frac{1}{2}\sum_{k=1}^{4}\prod_{\ell=1}^{8}\theta_k(\tau;m^\ell_{E_8}),$$

$$\eta(\tau) = \prod_{k=1}^{\infty}(1-Q_\tau^k), \quad \theta_1(z) = \eta(\tau)\prod_{k=1}^{\infty}(1-Q_\tau^k e^{2\pi i z})(1-Q_\tau^{k-1}e^{-2\pi i z}),$$

$$\theta_2(\tau;z) = \theta_1(\tau;z+1/2), \quad \theta_3(\tau;z) = \theta_1(\tau;z+1/2+\tau/2), \quad \theta_4(\tau;z) = \theta_1(\tau;z+\tau/2).$$

$$(Q_\tau = e^{2\pi i\tau})$$

Multiple heterotic strings

No bound states $\to$ n strings on $\mathbb{C}^4 \approx$ one string on $\mathrm{Sym}^n(\mathbb{C}^4)$.
Leads to a very elegant, simple formula [Dijkgraaf-Moore-Verlinde-Verlinde]

$$\sum_{n=0}^{\infty} e^{-nt} Z_n^H = \exp\left(\sum_{n=0}^{\infty} e^{-nt} T_n Z_1^H\right),$$

where

$$T_n f(\vec{z},\tau) = n^{k-1} \sum_{\substack{ad=n \\ a,d>0}} \frac{1}{d^k} \sum_{b \pmod{d}} f\left(a\vec{z}, \frac{a\tau+b}{d}\right) \qquad (n\text{-th Hecke transform}).$$

Simple physical interpretation: for example,

$$Z_2^H(\vec{z},\tau) = \frac{1}{2}\left[\underbrace{\left(Z_1^H(\vec{z},\tau)\right)^2}_{\text{two single-wound strings}} + \underbrace{Z_1^H(2\vec{z},2\tau) + Z_1^H(\vec{z},\tau/2) + Z_1^H(\vec{z},1/2+\tau/2)}_{\text{one string, winding twice around A, B, or A+B cycles}}\right]$$

No such simple expression is known for M- or E-strings, which form bound states.

Interlude: modular and Jacobi forms.

$f(\tau)$ is a modular form of weight k if

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL(2,\mathbb{Z}).$$

Example: Eisenstein series of weight $2k$,

$$E_{2k}(\tau)=\frac{1}{2\zeta(2k)}\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)}\frac{1}{(m+n\tau)^{2k}} \qquad (k>1).$$

$E_4(\tau), E_6(\tau)$ generate the polynomial ring of modular forms. Note, $E_2(\tau)$ is not modular because of an anomalous term:

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2E_2(\tau)-\pi i c(c\tau+d).$$

Quasi-modular forms $\equiv$ polynomials in $E_2(\tau), E_4(\tau), E_6(\tau)$.

Jacobi forms of one variable:

$$\phi\left(\frac{a\tau+b}{c\tau+d};\ \frac{z}{c\tau+d}\right)=(c\tau+d)^k e^{\frac{2\pi i c m z^2}{c\tau+d}}\phi(\tau;z)$$

$$(z = \text{elliptic parameter},\ k = \text{weight},\ m = \text{index})$$

Power series expansion:

$$\phi(\tau;z)=\sum_{p=0}^{\infty}\chi_p(\tau)\,z^p \qquad\qquad (\chi_p(\tau):\ \text{quasimodular forms}).$$

Modular anomaly: $\frac{1}{\phi(\tau;z)}\partial_{E_2}\phi(\tau;z)=-\frac{m}{12}z^2$.

$$\text{Example: } \theta_1(\tau;z)=\eta(\tau)^3 z\exp\left(-\sum_{k\geq 1}\overbrace{\frac{B_{2k}}{(2k)(2k)!}}^{\text{Bernoulli numbers}}E_{2k}(\tau)\,z^{2k}\right)$$

$$\partial_{E_2}\theta_1(\tau;z)=-\frac{1}{24}z^2\theta_1(\tau;z).$$

Generalization to many variables:

$$\phi\left(\frac{a\tau+b}{c\tau+d};\ \frac{\vec{z}}{c\tau+d}\right)=(c\tau+d)^k e^{\frac{2\pi ic\sum m_i z_i^2}{c\tau+d}}\phi(\tau;\vec{z})$$

Modular anomaly:

$$\frac{1}{\phi(\tau;\vec{z})}\partial_{E_2}\phi(\tau;\vec{z})=-\frac{1}{12}\left(\sum_i m_i z_i^2\right).$$

For example : $\partial_{E_2}\Theta_{E_8}(\tau;\vec{m}_{E_8})=-\frac{1}{24}\sum_{\ell=1}^{8}(m_{E_8}^{\ell})^2\Theta_{E_8}(\tau;\vec{m}_{E_8})$.

Modular anomaly and Hecke transform:

$$\frac{1}{T_n\phi(\tau;\vec{z})}\partial_{E_2}T_n\phi(\tau;\vec{z})=n\cdot\frac{1}{\phi(\tau;\vec{z})}\partial_{E_2}\phi(\tau;\vec{z})$$

Modular anomaly equation for heterotic strings

Since

$$Z_1^H = -\frac{\Theta_{E_8}(\tau; \vec{m}_{E_8,L})\Theta_{E_8}(\tau; \vec{m}_{E_8,R})}{\eta^{16}} \cdot \frac{\eta^4}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)},$$

we get:

$$\partial_{E_2} Z_1^H = -\frac{1}{24}\left(\sum_{\ell=1}^{8}((m^\ell_{E_{8,L}})^2 + (m^\ell_{E_{8,R}})^2) - \sum_{i=1}^{4}\epsilon_i^2\right) Z_1^H,$$

Modular anomaly equation for heterotic strings

Since

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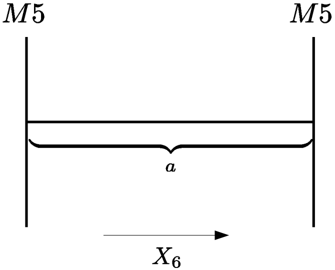
$$\partial_{E_2} Z_n^H = -\frac{n}{24}\left(\sum_{\ell=1}^{8}((m^{\ell}_{E_{8,L}})^2 + (m^{\ell}_{E_{8,R}})^2) - \sum_{i=1}^{4}\epsilon_i^2\right) Z_n^H.$$

M-strings

$$\underbrace{T^2_{0,1} \times \mathbb{C}^2_{2,3,4,5}}_{M5} \times \mathbb{R}_6 \times \mathbb{C}^2_{7,8,9,10}$$



$$\begin{array}{c} 6d\ \mathcal{N} = (2{,}0)\ A_1 \text{ theory on } \mathbb{R}^6 \\ \downarrow \quad \text{reduce on } S^1 \\ 5d\ \mathcal{N} = 2^*\, U(2) \text{ theory on } \mathbb{R}^5 \\ \downarrow \quad \text{compactify on } S^1 \\ 5d\ \mathcal{N} = 2^*\, U(2) \text{ theory on } (\mathbb{C}^2 \times S^1)_{\epsilon_1,\epsilon_2} \end{array}$$

Twists: $(z_1, z_2, z_3, z_4) \in \mathbb{C}^2_{2,3,4,5} \times \mathbb{C}^2_{7,8,9,10} \to (e^{2\pi i\epsilon_1} z_1, e^{2\pi i\epsilon_2} z_2, e^{2\pi i\epsilon_3} z_3, e^{2\pi i\epsilon_4} z_4)$.

a : Coulomb branch parameter;

$m = (\epsilon_3 - \epsilon_4)/2$: adj hypermultiplet mass;

$g^2_{YM} = \dfrac{2\pi i}{\tau}$: gauge coupling.

Usual instanton expansion à la Nekrasov gives:

$$Z^{5d}_{BPS}(g_{YM},\epsilon_1,\epsilon_2,m,a)=\sum_{n=0}^{\infty}e^{-4\pi n/g^2_{YM}}Z_{n\ \mathrm{inst}}(a,\epsilon_1,\epsilon_2,m).$$

In [1305.6322, 1310.1185] we obtained a very different expansion:

$$Z^{5d}_{U(2),BPS}(g_{YM},\epsilon_1,\epsilon_2,m,a)=\underbrace{(Z^{5d}_{U(1),BPS}(\tau,\epsilon_1,\epsilon_2,m))^2}_{\text{from } dof \text{ of individual M5 branes}}\cdot\underbrace{\sum_{n=0}^{\infty}e^{-n\,a}Z^{M}_{n}(\tau,\epsilon_1,\epsilon_2,m)}_{\text{M2 branes stretching between M5s}}\ !$$

Several dual descriptions for the 2d $\mathcal{N}=(2,0)$ worldsheet theory of n M-strings:

- SUSY sigma model on $Hilb_n(\mathbb{C}^2)$ (with right-moving fermions coupling to $E\oplus E^*$ instead of tangent bundle (E = tautological bundle)) [1305.6322];
- 2d U(N) SYM description [1310.1185];
- Reduction of ABJM [Hosomichi, Lee];
- Topological string description $\to$ QM description in terms of M5 domain wall operators [1305.6322]:

Open topological string amplitudes	=	M5 brane domain walls
$\downarrow$ glue		$\downarrow$ compute exp. value
Closed topological string amplitude	=	M-string elliptic genus

$$Z_{\text{top}}^{\text{closed}}=\sum_{\nu}e^{-|\nu|t}Z_{\text{top},\,\emptyset\nu}^{\text{open}}Z_{\text{top},\,\nu\emptyset}^{\text{open}}\equiv\sum_{\nu}e^{-|\nu|t}D_{\emptyset\nu}^{M5}D_{\nu\emptyset}^{M5}=\langle 0|\widehat{D}^{M5}e^{-\mathcal{H}t}\widehat{D}^{M5}|0\rangle$$

Domain wall formula for n M-string elliptic genus:

$$Z_n^M(m, \epsilon_1, \epsilon_2, \tau) = \sum_{|\nu|=n} D^{M5}_{\emptyset\nu} D^{M5}_{\nu\emptyset}$$

where $\widehat{D}^{M5}|0\rangle = \sum_\nu D^{M5}_{\nu\emptyset}|\nu\rangle$ are the M5 domain wall factors:

$$D^{\mathrm{M5}}_{\emptyset\nu} = \prod_{(i,j)\in\nu} \frac{\theta_1(\tau; -m + \epsilon_1(\nu_i - j + 1/2) - \epsilon_2(-i + 1/2))\eta(\tau)^{-1}}{\xi_-(\tau; \epsilon_1(\nu_i - j) - \epsilon_2(\nu^t_j - i + 1))\,\xi_+(\tau; \epsilon_1(\nu_i - j + 1) - \epsilon_2(\nu^t_j - i))},$$

$$D^{\mathrm{M5}}_{\nu\emptyset} = \prod_{(i,j)\in\nu} \frac{\theta_1(\tau; -m - \epsilon_1(\nu_i - j + 1/2) + \epsilon_2(-i + 1/2))\eta(\tau)^{-1}}{\xi_+(\tau; \epsilon_1(\nu_i - j) - \epsilon_2(\nu^t_j - i + 1))\,\xi_-(\tau; \epsilon_1(\nu_i - j + 1) - \epsilon_2(\nu^t_j - i))},$$

where $\quad \xi_+(\tau; z) = \prod_{k\geq 1}(1 - Q_\tau^k e^{2\pi i z}), \quad \xi_-(\tau; z) = \prod_{k\geq 1}(1 - Q_\tau^{k-1} e^{-2\pi i z}).$

Z_n^M should be $SL(2,\mathbb{Z})$-invariant, but $D^{M5}_{\emptyset\nu}, D^{M5}_{\nu\emptyset}$ are not good modular objects! However their combination is, since

$$\theta_1(\tau; z) = \eta(\tau)\xi_+(\tau; z)\xi_-(\tau; z).$$

From the expression for Z_n^M we can easily obtain the following modular anomaly equation for M-strings:

$$\frac{\partial Z_n^M}{\partial E_2} = -\frac{1}{12}\left(\epsilon_1\epsilon_2 n^2 + n(m^2 - (\epsilon_+/2)^2)\right) Z_n^M$$

where $\epsilon_+ = \epsilon_1 + \epsilon_2$.

E-strings: known facts

Twists: $(z_1, z_2, z_3, z_4) \in \mathbb{C}^2_{2,3,4,5} \times \mathbb{C}^2_{7,8,9,10} \to (e^{2\pi i \epsilon_1} z_1, e^{2\pi i \epsilon_2} z_2, e^{2\pi i \epsilon_3} z_3, e^{2\pi i \epsilon_4} z_4)$.

Worldsheet theory of n E-strings has $(2,0)$ SUSY, couples to level n $\widehat{E}_8$ current algebra [Minahan-Nemeschansky-Vafa-Warner].

Elliptic genus depends on $(\tau, \epsilon_1, \epsilon_2, \vec{m}_{E_8,L})$, not on $(\epsilon_3 - \epsilon_4)/2$ – R-symmetry of (1,0) SCFT is too small to accommodate for it. Also, fermions that would have coupled to it are replaced by internal bosons compactified on Γ_{E_8}.

One E-string elliptic genus: $$Z_1^E = -\underbrace{\frac{\Theta_{E_8}(\tau; \vec{m}_{E_8,L})}{\eta^8}}_{\text{8 bosons on } \Gamma_{E_8} \text{ lattice}} \cdot \underbrace{\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)}}_{\text{4 spacetime bosons}}.$$

E-strings form bound states $\rightarrow$ not known *a priori* how to compute Z_n^E from Z_1^E.

Partial results from duality with topological strings [Klemm-Mayr-Vafa, Minahan-Nemeschansky-Vafa-Warner, Hosono-Saito-Takahashi, Mohri, Eguchi-Sakai, Iqbal, Sakai, Huang-Klemm-Poretschkin,...]:

$$\mathcal{N} = (1,0) \text{ theory on } \mathbb{R}^6$$
$$\downarrow$$
$$5d \;\; \text{theory on } (\mathbb{R}^4 \times S^1)_{\epsilon_1,\epsilon_2}$$
$$\downarrow \text{ [Morrison-Vafa]}$$

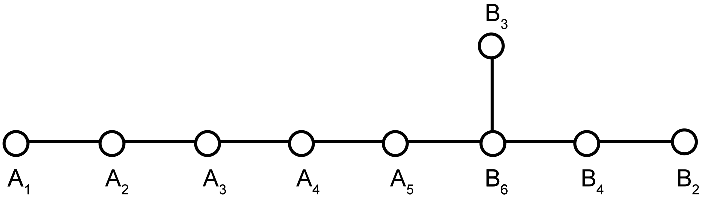
Refined topological strings on $X = \mathcal{O}(-K) \longrightarrow \frac{1}{2}K3$
(anti-canonical bundle over the'half-K3' surface $\frac{1}{2}K3 = dP_9$).

$$Z_{top}(X;\epsilon_1,\epsilon_2) \equiv Z^{5d}_{BPS} = \sum_{n=0}^{\infty} e^{-nt} Z^E_n(\epsilon_1,\epsilon_2,\vec{m}_{E_8,L},\tau)$$

$(t,\tau,\vec{m}_{E_8,L}) =$ Kähler parameters of X

Can write $Z_{top} = \exp\left(\sum_{n\geq 0, g\geq 0, \ell\geq 0} e^{-nt}(-\epsilon_1\epsilon_2)^{g-1}\,\epsilon_+^{2\ell}\,\mathcal{F}_{n,g,\ell}(\tau,\vec{m}_{E_8,L})\right)$ and compute $\mathcal{F}_{n,g,\ell}$, for any given n, as perturbative expansion up to some power of g and ℓ [Huang-Klemm-Poretschkin].

$\mathcal{F}_{n,g,\ell}$ can be written in terms of modular invariant combinations of $\widehat{E}_8$ characters



$$A_1(\vec{m}_{E_8,L},\tau) = \Theta_{E_8}(m_{E_8,L},\tau) \quad \text{(level 1 character of affine } E_8\text{);}$$

$$A_n(\vec{m}_{E_8,L},\tau) = \frac{n^3}{n^3+1} T_n(A_1) \qquad \text{(Hecke transform of } A_1\text{);}$$

B_n = more complicated functions...

Note also that A_n (B_n) have modular weight 4 (6), and

$$\partial_{E_2} A_n = \tfrac{n}{12} A_n, \quad \partial_{E_2} B_n = \tfrac{n}{12} B_n.$$

For example, for 1 E-string:

$$\mathcal{F}_{1,0,0} = \frac{A_1}{\eta^{12}}; \quad \mathcal{F}_{1,1,0} = -\frac{E_2}{24}\frac{A_1}{\eta^{12}}; \quad \mathcal{F}_{1,0,1} = \frac{E_2}{12}\frac{A_1}{\eta^{12}}; \quad \ldots$$

For 2 E-strings:

$$\mathcal{F}_{2,0,0} = \frac{4E_2A_1^2 + 3E_6A_2 + 5E_4B_2}{96\eta^{24}};$$

$$\mathcal{F}_{2,1,0} = -\frac{4E_2^2A_1^2 + 4A_1^2E_4 + 3A_2E_4^2 + 5E_2E_4B_2 + 3A_2E_2E_6 + 5B_2E_6}{1152\eta^{24}}$$

$$\mathcal{F}_{2,0,1} = \frac{10A_1^2E_2^2 + 6A_1^2E_4 + 9A_2E_2E_6 + 3A_2E_4^2 + 15B_2E_2E_4 + 5B_2E_6}{1152\eta^{24}}$$

$$\ldots$$

$(E_2, E_4, E_6 = \text{Eisenstein series})$

E-string modular anomaly (with $\vec{m}_{E_8,L}=0$) [Hosono-Saito-Takahashi, Huang-Klemm-Poretschkin]:

$$\begin{aligned}\partial_{E_2}\mathcal{F}_{n,g,\ell} &= \frac{1}{24}\sum_{\nu=1}^{n-1}\sum_{\gamma=0}^{g}\sum_{\lambda=0}^{\ell}\nu(n-\nu)\mathcal{F}_{\nu,\gamma,\lambda}\mathcal{F}_{n-\nu,g-\gamma,\ell-\lambda}\\ &+ \frac{n(n+1)}{24}\mathcal{F}_{n,g-1,\ell}-\frac{n}{24}\mathcal{F}_{n,g,\ell-1}.\end{aligned}$$

For the n E-string elliptic genus (with $\vec{m}_{E_8,L}$ dependence restored):

$$\partial_{E_2}Z_n^E=-\frac{1}{24}\left[\epsilon_1\epsilon_2(n^2+n)-\epsilon_+^2 n+n\left(\sum_{i=1}^{8}m_{E_8,i}^2\right)\right]\cdot Z_n^E.$$

M9 domain walls

Claim: E-strings and heterotic strings can also be written in terms of domain walls:

$$Z_n^E = \sum_\nu D_\nu^{M9,L} D_{\nu\emptyset}^{M5}, \qquad Z_n^H \sim \sum_\alpha D_\alpha^{M9,L} D_\alpha^{M9,R}$$

Recall:

$$D_{\nu\emptyset}^{\mathrm{M5}} = \prod_{(i,j)\in\nu} \frac{\theta_1(\tau; -m - \epsilon_1(\nu_i - j + 1/2) + \epsilon_2(-i+1/2))\eta(\tau)^{-1}}{\xi_+(\tau; \epsilon_1(\nu_i - j) - \epsilon_2(\nu_j^t - i + 1))\,\xi_-(\tau; \epsilon_1(\nu_i - j + 1) - \epsilon_2(\nu_j^t - i))}$$

$$= \frac{F_\nu^R(m, \epsilon_1, \epsilon_2, \tau)}{B_\nu^R(\epsilon_1, \epsilon_2, \tau)} \qquad \begin{matrix} \leftarrow \text{`fermionic pieces'} \\ \leftarrow \text{`bosonic pieces'} \end{matrix}$$

$$D_{\emptyset\nu}^{\mathrm{M5}} = \prod_{(i,j)\in\nu} \frac{\theta_1(\tau; -m + \epsilon_1(\nu_i - j + 1/2) - \epsilon_2(-i+1/2))\eta(\tau)^{-1}}{\xi_-(\tau; \epsilon_1(\nu_i - j) - \epsilon_2(\nu_j^t - i + 1))\,\xi_+(\tau; \epsilon_1(\nu_i - j + 1) - \epsilon_2(\nu_j^t - i))}$$

$$= \frac{F_\nu^L(m, \epsilon_1, \epsilon_2, \tau)}{B_\nu^L(\epsilon_1, \epsilon_2, \tau)} \qquad \begin{matrix} \leftarrow \text{`fermionic pieces'} \\ \leftarrow \text{`bosonic pieces'} \end{matrix}$$

E-string elliptic genus is modular, m-independent $\rightarrow$ make the Ansatz

$$D_\nu^{M9,L} \propto \frac{1}{F_\nu^R(m,\epsilon_1,\epsilon_2,\tau)B_\nu^L(\epsilon_1,\epsilon_2,\tau)}, \quad D_\nu^{M9,R} \propto \frac{1}{F_\nu^L(m,\epsilon_1,\epsilon_2,\tau)B_\nu^R(\epsilon_1,\epsilon_2,\tau)}.$$

Including $\widehat{E}_8$ contribution, we guess:

$$D_\nu^{M9,L} = \frac{N_\nu(\vec{m}_{E_8,L},\epsilon_1,\epsilon_2,\tau)}{\eta^{8n}F_\nu^R(m,\epsilon_1,\epsilon_2,\tau)B_\nu^L(\epsilon_1,\epsilon_2,\tau)},$$

$$D_\nu^{M9,R} = \frac{N_\nu(\vec{m}_{E_8,R},\epsilon_1,\epsilon_2,\tau)}{\eta^{8n}F_\nu^R(m,\epsilon_1,\epsilon_2,\tau)B_\nu^L(\epsilon_1,\epsilon_2,\tau)}.$$

Elliptic genus has modular weight 0 $\rightarrow$ $N_\nu(\vec{m}_{E_8},\epsilon_1,\epsilon_2,\tau)$ has weight $4n$.

E-string modular anomaly

Assume $Z_n^M \sim \sum D^{M5} \cdot D^{M5}$ and $Z_n^H \sim \sum D^{M9} \cdot D^{M9}$, and modular anomaly equation holds term by term. Set $\vec{m}_{E_8,R} = \vec{m}_{E_8,L}$, and note that

M-string : $$D^{M5} \cdot D^{M5} \sim \frac{F^L \cdot F^R}{B^L \cdot B^R},$$

H-string : $$D^{M9} \cdot D^{M9} \sim \frac{N(\vec{m}_{E_8,L})^2}{(F^L \cdot F^R)(B^L \cdot B^R)},$$

E-string : $$D^{M9} \cdot D^{M5} \sim \frac{N(\vec{m}_{E_8,L})}{B^L \cdot B^R} = \sqrt{(D^{M5} \cdot D^{M5}) \cdot (D^{M9} \cdot D^{M9})}!$$

Leads to prediction:

$$\text{E-string mod. anomaly} = \frac{1}{2}\Big[(\text{M-string mod. anomaly}) + (\text{H-string mod-anomaly})\Big].$$

Checked easily, by explicit computation!

One string elliptic genus

This is simple: recall that

$$Z_1^E = -\frac{A_1(\vec{m}_{E_8,L},\tau)}{\eta^8}\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} = D_\square^{M9,L} D_{\square\emptyset}^{M5}.$$

Since $D_{\square\emptyset}^{M5} = \dfrac{\theta_1(-m-(\epsilon_1+\epsilon_2)/2)\eta^{-1}}{\xi_-(\epsilon_1)\xi_+(-\epsilon_2)}$, we get

$$\begin{aligned}
D_\square^{M9,L} &= \left(\frac{A_1(\vec{m}_{E_8,L})}{\eta^8}\right)\frac{\eta}{\theta_1(-m-\epsilon_+/2)}\cdot\frac{1}{\xi_+(\epsilon_1)\xi_-(-\epsilon_2)}\\
&= \left(\frac{A_1(\vec{m}_{E_8,L})}{\eta^8}\right)\frac{\eta}{\theta_1(\epsilon_3)}\cdot\frac{1}{\xi_+(\epsilon_1)\xi_-(-\epsilon_2)}.\\
D_\square^{M9,R} &= \left(\frac{A_1(\vec{m}_{E_8,R})}{\eta^8}\right)\frac{\eta}{\theta_1(\epsilon_4)}\cdot\frac{1}{\xi_-(\epsilon_1)\xi_+(-\epsilon_2)}.
\end{aligned}$$

One string elliptic genus

This is simple: recall that

$$Z_1^E = -\frac{A_1(\vec{m}_{E_8,L},\tau)}{\eta^8}\frac{\eta^2}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)} = D_{\square}^{M9,L} D_{\square\emptyset}^{M5}.$$

Since $D_{\square\emptyset}^{M5} = \frac{\theta_1(-m-(\epsilon_1+\epsilon_2)/2)\eta^{-1}}{\xi_-(\epsilon_1)\xi_+(-\epsilon_2)}$, we get

$$\begin{aligned} D_{\square}^{M9,L} &= \left(\frac{A_1(\vec{m}_{E_8,L})}{\eta^8}\right)\frac{\eta}{\theta_1(-m-\epsilon_+/2)}\cdot\frac{1}{\xi_+(\epsilon_1)\xi_-(-\epsilon_2)} \\ &= \left(\frac{A_1(\vec{m}_{E_8,L})}{\eta^8}\right)\frac{\eta}{\theta_1(\epsilon_3)}\cdot\frac{1}{\xi_+(\epsilon_1)\xi_-(-\epsilon_2)}. \\ D_{\square}^{M9,R} &= \left(\frac{A_1(\vec{m}_{E_8,R})}{\eta^8}\right)\frac{\eta}{\theta_1(\epsilon_4)}\cdot\frac{1}{\xi_-(\epsilon_1)\xi_+(-\epsilon_2)}. \end{aligned}$$

Check:

$$D_{\square}^{M9,L} D_{\square}^{M9,R} = -\left(\frac{A_1(\vec{m}_{E_8,L})\times A_1(\vec{m}_{E_8,R})}{\eta^{16}}\right)\frac{\eta^4}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)} = Z_1^{het}!$$

Two string elliptic genus

$$Z_2^E \stackrel{?}{=} D^{M9,L}_{\Box\Box} D^{M5}_{\Box\Box\emptyset} + D^{M9,L}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} D^{M5}_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}\emptyset}$$

Need to determine E_8 factors $N_{\Box\Box}$, $N_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}$ (of modular weight 8).

- Modular anomaly equation $\rightarrow N_{\Box\Box} = N(\vec{m}_{E_8,L}, \epsilon_1, \tau)$.
- $(\epsilon_1 \leftrightarrow \epsilon_2)$ symmetry $\rightarrow N_{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}} = N(\vec{m}_{E_8,L}, \epsilon_2, \tau)$.
- $\widehat{E}_8$ level $= 2$

So we make the Ansatz:

$$N(\vec{m}_{E_8,L}, \epsilon, \tau) = A_1(\vec{m}_{E_8,L})^2 f_0(\tau;\epsilon) + B_2(\vec{m}_{E_8,L}) f_2(\tau;\epsilon) + A_2(\vec{m}_{E_8,L}) f_4(\tau;\epsilon)$$

($f_0, f_2, f_4 =$ holomorphic Jacobi forms of weight 0, 2, 4).

Functions $f_{0,2,4}$ are uniquely fixed by comparing with $\mathcal{F}_{2,g,\ell}$ for low values of g, ℓ.

We find:

$$N(\vec{m}_{E_8,L},\epsilon,\tau) = \frac{1}{576}\Big[4A_1^2(\phi_{0,1}(\epsilon)^2 - E_4\phi_{-2,1}(\epsilon)^2)$$
$$+ 3A_2(E_4^2\phi_{-2,1}(\epsilon)^2 - E_6\phi_{-2,1}(\epsilon)\phi_{0,1}(\epsilon)) + 5B_2(E_6\phi_{-2,1}(\epsilon)^2 - E_4\phi_{-2,1}(\epsilon)\phi_{0,1}(\epsilon))\Big],$$

where

$$\phi_{-2,1}(\epsilon,\tau) = -\frac{\theta_1(\epsilon;\tau)^2}{\eta^6(\tau)}, \qquad \phi_{0,1}(\epsilon,\tau) = 4\left[\frac{\theta_2(\epsilon;\tau)^2}{\theta_2(0;\tau)^2} + \frac{\theta_3(\epsilon;\tau)^2}{\theta_3(0;\tau)^2} + \frac{\theta_4(\epsilon;\tau)^2}{\theta_4(0;\tau)^2}\right].$$

Putting everything together, we get the formula:

$$Z_2^{\text{E-str}} = -\frac{N(\vec{m}_{E_8,L},\epsilon_1,\tau)/\eta^{16}}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_1-\epsilon_2)\theta_1(2\epsilon_1)\eta^{-4}} - \frac{N(\vec{m}_{E_8,L},\epsilon_2,\tau)/\eta^{16}}{\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_2-\epsilon_1)\theta_1(2\epsilon_2)\eta^{-4}}$$

in perfect agreement with [Huang-Klemm-Poretschkin]! Note: $Z_2^E \neq T_2 Z_1^E$, as expected.

How about 2 heterotic strings?

$$Z_2^H \stackrel{?}{=} D^{M9,L}_{\square\square}(\vec{m}_{E_8,L}) D^{M9,R}_{\square\square}(\vec{m}_{E_8,R}) + D^{M9,L}_{\substack{\square\\\square}}(\vec{m}_{E_8,L}) D^{M9,R}_{\substack{\square\\\square}}(\vec{m}_{E_8,R})$$

$$= -\frac{N(\vec{m}_{E_8,L},\epsilon_1,\tau)N(\vec{m}_{E_8,R},\epsilon_1,\tau)}{\eta(\tau)^{24}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)\theta_1(2\epsilon_1)\theta_1(\epsilon_1-\epsilon_2)\theta_1(\epsilon_1-\epsilon_3)\theta_1(\epsilon_1-\epsilon_4)}$$

$$+(\epsilon_1 \leftrightarrow \epsilon_2)$$

This doesn't work (not even invariant under permutations of $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$)!
However, the following works:

$$Z_2^H = -\frac{N(\vec{m}_{E_8,L},\epsilon_1,\tau)N(\vec{m}_{E_8,R},\epsilon_1,\tau)}{\eta(\tau)^{24}\theta_1(\epsilon_1)\theta_1(\epsilon_2)\theta_1(\epsilon_3)\theta_1(\epsilon_4)\theta_1(2\epsilon_1)\theta_1(\epsilon_1-\epsilon_2)\theta_1(\epsilon_1-\epsilon_3)\theta_1(\epsilon_1-\epsilon_4)}$$

$$+(\epsilon_1 \leftrightarrow \epsilon_2) + (\epsilon_1 \leftrightarrow \epsilon_3) + (\epsilon_1 \leftrightarrow \epsilon_4)$$

$$\stackrel{!!}{=} T_2 Z_1^H.$$

Very nontrivial! Checked up to high powers of $e^{2\pi i\tau}$, with arbitrary $E_8 \times E_8$ twists.

Summary

- Can define (and compute, at least up to 2 strings) a QM state $|M9\rangle$ for M2 branes ending on an M9.
- This leads to the correct E-string modular anomaly, given M, H modular anomaly.
- Z_1^H and Z_1^E are easily expressed in terms of M9 domain walls.
- M9 domain walls lead to an exact formula for Z_2^E + a new formula for Z_2^{het}.

Some open questions:

- Need a better understanding of M9 boundary conditions. Would be interesting to study E-strings from ABJM theory along lines of [Hosomichi,Lee].
- Can one compute $D_{\mu\nu}^{M5}$, D_{α}^{M9} directly from ABJM theory reduced on a T^2?
- Open topological string computation of $|M9\rangle$? There are some hints this may be the case (integral expansion...)

$$N(\vec{m}_{E_8,L},\epsilon,\tau) = A_1(\vec{m}_{E_8,L})^2 f_0(\tau;\epsilon) + B_2(\vec{m}_{E_8,L}) f_2(\tau;\epsilon) + A_2(\vec{m}_{E_8,L}) f_4(\tau;\epsilon).$$

The polynomial ring of weak holomorphic Jacobi forms with modular parameter τ and elliptic parameter ϵ of index k and even weight w is generated by the four modular forms $E_4(\tau), E_6(\tau), \phi_{0,1}(\tau;z)$, and $\phi_{-2,1}(\tau;z)$, where

$$\phi_{-2,1}(\tau;z) = -\frac{\theta_1(\tau;z)^2}{\eta^6(\tau)} \qquad \text{and} \qquad \phi_{0,1}(\tau;z) = 4\left[\frac{\theta_2(\tau;z)^2}{\theta_2(\tau;0)^2} + \frac{\theta_3(\tau;z)^2}{\theta_3(\tau;0)^2} + \frac{\theta_4(\tau;z)^2}{\theta_4(\tau;0)^2}\right]$$

are Jacobi forms of index 1, respectively of weight -2 and 0.

Thus modularity implies that f_1, f_2, f_3 can be written as follows:

$$f_0(\epsilon) = c_{0,1}\phi_{0,1}(\epsilon)^2 + c_{0,2}E_4\phi_{-2,1}(\epsilon)^2; \tag{1}$$

$$f_2(\epsilon) = c_{2,1}E_4\phi_{0,1}(\epsilon)\phi_{-2,1}(\epsilon) + c_{2,2}E_6\phi_{-2,1}(\epsilon)^2; \tag{2}$$

$$f_4(\epsilon) = c_{4,1}E_4\phi_{0,1}(\epsilon)^2 + c_{4,2}E_6\phi_{0,1}(\epsilon)\phi_{-2,1}(\epsilon) + c_{4,3}E_4^2(\tau)\phi_{-2,1}(\epsilon)^2. \tag{3}$$

Coefficients can be fixed by comparing Z_E^1 with $\mathcal{F}_2 + \frac{1}{2}\mathcal{F}_1^2$ (computed from topological strings).

Three strings

Requires more work, but we can make a simple check: to order Q_τ^0,

$$\theta_1(z,\tau) \sim (1 - e^{2\pi i z}), \text{ and we take } N_{\square\square\square} \sim N_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}} \sim N_{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}} \sim 1 + \mathcal{O}(Q_\tau).$$

From orbifold formula (taking $q_k = e^{2\pi i \epsilon_k}$),

$$Z_3^H(\vec{\epsilon}) = -\frac{1}{6}\left[\frac{1}{\prod_{k=1}^4 (1-q_k)^3} + 3\frac{1}{\prod_{k=1}^4 (1-q_k)(1-q_k^2)} + 2\frac{1}{\prod_{k=1}^4 (1-q_k^3)}\right]$$

Three strings

Requires more work, but we can make a simple check: to order Q_τ^0,

$$\theta_1(z,\tau) \sim (1 - e^{2\pi i z}), \text{ and we take } N_{\yng(3)} \sim N_{\yng(2,1)} \sim N_{\yng(1,1,1)} \sim 1 + \mathcal{O}(Q_\tau).$$

From orbifold formula (taking $q_k = e^{2\pi i \epsilon_k}$),

$$Z_3^H(\vec{\epsilon}) = -\frac{1}{6}\left[\frac{1}{\prod_{k=1}^4 (1-q_k)^3} + 3\frac{1}{\prod_{k=1}^4 (1-q_k)(1-q_k^2)} + 2\frac{1}{\prod_{k=1}^4 (1-q_k^3)}\right]$$

From domain wall formula we guess:

$$-\left[\frac{q_1^6}{\left(\prod_{k=1}^4 (1-q_1)\right)(1-q_1^2)(1-q_1^3)\left(\prod_{k=2}^4 (q_1-q_k)(q_1^2-q_k)\right)} + (3 \text{ permutations})\right]$$

$$-\left[\frac{q_1^2 q_2^2}{\left(\prod_{k=1}^4 (1-q_k)\right)(1-q_1)(1-q_2)(q_1-q_2^2)(q_2-q_1^2)\left(\prod_{k=3}^4 (q_1-q_k)(q_2-q_k)\right)} + (5 \text{ permutations})\right]$$

The two expressions agree!