Supersymmetric backgrounds from 5d $\mathcal{N}=1$ supergravity

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Introduction

Recent progress in SUSY gauge theories

→ Exact results

- Partition function, Indices
- Wilson loops, Polyakov loops
- Surface operators
- Domain walls

By using localization we can compute these quantities exactly.

(Supersymmetric) localization

$$Z = \int \mathcal{D}\Phi \, e^{-S_0[\Phi]} \qquad (QS_0 = 0)$$

$\downarrow$ Q-exact deformation

$$Z(t) = \int \mathcal{D}\Phi \, e^{-S_0 - tQV} \qquad \begin{pmatrix} t \in \mathbb{R}_+ & QV \geq 0 \\ Q^2 V = 0 & \end{pmatrix}$$

This is independent of t.

We can perform the path integral in $t \to \infty$ limit.

Many results has been obtained for various background manifolds

3d and 4d examples

T^4 : Witten('82) $\Longrightarrow$ Witten Index

S^4 : Pestun (0712.2824)

S^3 : Kapustin, Willett, Yaarkov (0909.4559)

$\vdots$

These results are based on the realization of rigid supersymmetry Q on curved backgrounds.

A systematic way of constructing SUSY theories on curved backgrounds using <u>off-shell supergravity</u> was proposed in Festuccia - Seiberg (1105.0689)

Weyl multiplet $\left(e_\mu^{\hat{\nu}},\ \underbrace{\psi_{\mu\alpha}}_{\text{gravitino}},\ \cdots \right)$

Rigid SUSY

$$\delta_Q(\xi)\psi_\mu \equiv D_\mu \xi + \cdots = 0$$

$\delta_Q(\xi)$: local supersymmetry

Integrability condition $\rightarrow$ SUSY backgrounds

Supersymmetric backgrounds in 4d and 3d

Klare-Tomasiello-Zaffaroni (1205.1069)

Dumitrescu-Festuccia-Seiberg (1205.1115)

Dumitrescu-Festuccia (1209.5408)

Closset-Dumitrescu-Festuccia-Komargodski (2013.1212)

Cassani-Klare-Martelli-Tomasiello-Zaffaroni (1207.2181)

⋮

We also had $\left\{ \begin{array}{l} \text{Martelli's talk} \\ \text{Sparks' talk} \end{array} \right.$ on Tuesday

These analyses showed that the constraints are not so strong.

There are infinitely many deformations that preserve supersymmetry.

(functional degrees of freedom)

Q: How does Z depend on the deformation parameters?

A: Only a few parameters affect Z (in 3d and 4d)

This is because almost all deformations can be realized as <u>Q - exact deformations</u>.

$S_0 \xrightarrow{\text{deform}} S_0 + S_1 \qquad S_1 = QV + (\text{a few exceptions})$

3d and 4d

Closset - Dumitrescu - Festuccia - Komargodski
1309.5876

$\Rightarrow$ Let us perform similar analysis in <u>5d</u>.

- Solution to $\delta_Q(\text{fermions}) = 0$
- Deformation dependence of Z

Plan

- ✓ Introduction
- 5d $\mathcal{N}=1$ off-shell supergravity
- Supersymmetric backgrounds
- Deformation dependence of Z
- Summary

5d $\mathcal{N}=1$ off-shell supergravity

$\left\{ \begin{array}{l} \text{Zuker '99} \\ \text{Kugo-Ohashi '00} \end{array} \right.$

5d $\mathcal{N}=1$ super-Poincare algebra

$P_{\hat{\mu}}$: translation (diffeomorphism)

$Q_{\alpha I}$: supersymmetry

Z : central charge $U(1)_Z$

$M_{\hat{\mu}\hat{\nu}}$: $Sp(2)_r \sim SO(5)_r$ rotation (Euclidean)

R_a : $Sp(1)_R \sim SU(2)_R$

indices

$\mu, \nu, \cdots = 1,2,3,4,5$	global indices	
$\hat{\mu}, \hat{\nu}, \cdots = \hat{1},\hat{2},\hat{3},\hat{4},\hat{5}$	local indices	
$\alpha, \beta, \cdots = 1,2,3,4$	spinor indices	
$I, J, \cdots = 1,2$	$Sp(1)_R$ doublet	
$a, b, \cdots = 1,2,3$	$Sp(1)_R$ triplet	

$$\{Q, Q\} \sim \underbrace{\overbrace{P + Z}^{\text{flat background}} + M + R}_{\text{curved background}}$$

Gravity (Weyl) multiplet includes "gauge fields"
for these generators.

			$SO(5)_r$	$SU(2)_R$	$U(1)_Z$
generators	$P_{\hat{\mu}}$,	$Q_{\alpha I}$,	$M_{\hat{\mu}\hat{\nu}}$,	R_a,	Z
currents	$T^{\nu}_{\hat{\mu}}$,	$S^{\mu}_{\alpha I}$,		R^{μ}_a,	J^{μ} $\in$ supercurrent multiplet
gauge fields	$e^{\hat{\mu}}_{\nu}$,	$\psi_{I\mu\alpha}$,	$(\omega_{\lambda\hat{\mu}\hat{\nu}})$,	V^a_{μ},	a_{μ} $\in$ Weyl multiplet.
d.o.f.	10_B	32_F		12_B	4_B

bosonic fermionic

$10+12+4=26 \neq 32$

Additional fields are needed.

What kinds of additional fields should we introduce?

The structure of the Weyl multiplet
can be determined from the supercurrent multiplet.

Supercurrent multiplet

additional operators.

$$\left(T_{\hat{\mu}}^{\ \nu},\ S^{\mu}_{I\alpha},\ R^{\mu}_{a},\ J^{\mu},\ \cdots \right)$$

Supersymmetric Yang-Mills theory

vector multiplet $(A_\mu, \lambda_{I\alpha}, \phi, D_a)$

SUSY tr. on the flat $\mathbb{R}^5$

$$\delta_Q(\xi)A_\mu = -(\xi\gamma_\mu\lambda)$$

$$\delta_Q(\xi)\lambda = -\not{F}\xi + i(\not{D}\phi)\xi + iD_a\tau_a\xi$$

$$\delta_Q(\xi)\phi = i(\xi\lambda)$$

$$\delta_Q(\xi)D_a = i(\xi\tau_a\not{D}\lambda) - i(\xi\tau_a[\phi,\lambda])$$

Invariant action on the flat $\mathbb{R}^5$

$$S_0 = \int d^5x\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi D^\mu\phi - \frac{1}{2}(\lambda\not{D}\lambda) - \frac{1}{2}D_aD_a - \frac{1}{2}\lambda[\phi,\lambda]\right)$$

Noether current for δ_Q (supersymmetry current)

$$S^\mu = -\not{F}\gamma^\mu\lambda + i(\not{\partial}\phi)\gamma^\mu\lambda \qquad (32_F)$$

E-M tensor
$$T^{\mu\nu} = F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \cdots \qquad (10_B)$$

$SU(2)_R$ current
$$R^\mu_a = \frac{1}{2}(\lambda\tau_a\gamma^\mu\lambda) \qquad (12_B)$$

$U(1)_Z$ current
$$J^\mu = \frac{1}{4}\epsilon^{\mu\kappa\lambda\rho\sigma}F_{\kappa\lambda}F_{\rho\sigma} - 2i\partial_\nu(F^{\mu\nu}\phi) \qquad (4_B)$$

These operators does not form a closed rep of δ_Q.

repetition of δ_Q gives

$$\begin{cases} M_{\rho\sigma} = iF_{\rho\sigma}\phi + \frac{1}{8}(\lambda\gamma_{\rho\sigma}\lambda) & (10_B) \\ \Phi = \frac{1}{2}\phi^2 & (1_B) \\ \chi_I = i\lambda_I\phi & (8_F) \\ X_a = \frac{1}{2}(\lambda\tau_a\lambda) & (3_B) \end{cases}$$

We obtained supercurrent multiplet with $40_B + 40_F$ d.o.f.

$$\Big(\underbrace{T^{\mu}_{\hat{\nu}},\, S^{\mu}_{I\alpha},\, R^{\mu}_{a},\, J^{\mu}}_{\text{conserved currents}},\, \underbrace{M_{\mu\nu},\, \Phi,\, X_{I\alpha},\, X_a}_{\text{additional operators}} \Big)$$

Weyl multiplet with $40_B + 40_F$ d.o.f.

$$\Big(\underbrace{e^{\hat{\nu}}_{\mu},\, \psi_{\mu I\alpha},\, V^{a}_{\mu},\, a_{\mu}}_{\text{gauge fields}},\, \underbrace{v^{\mu\nu},\, C,\, \eta_{I\alpha},\, t^{a}}_{\text{additional fields}} \Big)$$

additional fields $14_B + 8_F$

$v_{\mu\nu}$	anti-sym tensor	10_B
t_a	triplet scalars	3_B
$\eta_{I\alpha}$	fermion	8_F
C	singlet scalar	1_B

matter action in a weakly curved background

linear in the Weyl mult.

$$S = \underset{\text{flat}}{S_0} + \underset{\text{small deformation}}{S_1}$$

$$S_1 = \int d^5x \left[-\Delta e_\mu^{\hat{\nu}} T_{\hat{\nu}}^{\mu} + \Delta \psi_\mu S^\mu - \Delta V_\mu^a R_a^\mu - \Delta a_\mu J^\mu + \cdots \right]$$

For the following analysis we need the full (non-linear) action

Construction is tedious.

Fortunately it is given in

Kugo-Ohashi $\begin{cases} \text{hep-ph/0006231} \\ \text{hep-ph/0010288} \end{cases}$

Supersymmetric backgrounds

Once a background Weyl multiplet is given we can give the susy action of matter fields on the background.

For the existence of rigid susy $\delta_Q(\xi)$, the background must satisfy

$$\boxed{\delta_Q(\xi)\,(\text{fermions}) = 0}$$

There are two fermions $\psi_{\mu\alpha I}$ and $\eta_{\alpha I}$ in the Weyl mult.

We should consider two conditions.

Susy backgrounds are defined by

$$\begin{cases} \delta_Q(\xi)\psi_\mu = 0 & \text{Killing spinor eq.} \\ \delta_Q(\xi)\eta = 0 & \text{Auxiliary eq.} \end{cases}$$

Susy tr.

$$\delta_Q(\xi)\psi_\mu = D_\mu \xi - f_{\mu\nu}\gamma^\nu \xi + \frac{1}{4}\gamma_{\mu\rho\sigma}v^{\rho\sigma}\xi - t\gamma_\mu \xi$$

$$\delta_Q(\xi)\eta = -2\gamma_\nu \xi D_\mu v^{\mu\nu} + \xi C + 4(\cancel{\partial} t)\xi + 8(\cancel{f} - \cancel{v})t\xi + \gamma^{\mu\nu\rho\sigma}\xi f_{\mu\nu} f_{\rho\sigma}$$

($f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$: $U(1)_Z$ field strength)

Comment on the parameter ξ

Lorenzian case : $\xi \sim \xi^*$ (symplectic Majorana)

Euclidean case : ξ, ξ^* independent.

Although we consider Euclidean case,
we impose the symplectic Majorana condition

$$\xi_{\alpha I} = J_{\alpha\beta}\,\epsilon_{IJ}\,(\xi_{\beta J})^*$$

just for simplicity.

((2,4) of $SU(2)_R \times SO(5) = Sp(1)_R \times Sp(2)$ is real) non-vanishing bi-linears

$$S = (\bar{\xi}\xi), \quad R^{\mu} = (\bar{\xi}\gamma^{\mu}\xi), \quad J^{a}_{\mu\nu} = \frac{1}{S}(\bar{\xi}\tau_a\gamma_{\mu\nu}\xi)$$

ξ is real and nowhere vanishing $\rightarrow$ $S > 0$

Fierz identity $\rightarrow$ $R^{\mu}R_{\mu} = S^2 > 0$

$\rightarrow$ R^{μ} define a foliation

We can locally define x^5 by

$$R^{\mu}\partial_{\mu} = \partial_5$$

and take the local frame such that

$$e^{\hat{m}} = e^{\hat{m}}_{n}\, dx^{n}, \quad e^{\hat{5}} = S(dx^5 + V_m dx^m)$$

$$ds^2 = g_{mn}\, dx^m dx^n + S^2(dx^5 + V_m dx^m)^2$$

$\left(\begin{array}{l} m, n = 1,2,3,4 \\ \hat{m}, \hat{n} = \hat{1},\hat{2},\hat{3},\hat{4} \end{array}\right)$

$e^{\hat{m}}_{n}$, S, V_m may depend on x^5

Tangent space is divided into

$$\begin{cases} \text{horizontal space} & \hat{m} = \hat{1}, \hat{2}, \hat{3}, \hat{4} \\ \text{vertical space} & \hat{5} \end{cases}$$

$J^a_{\mu\nu} R^\nu = 0 \;\Rightarrow\;$ $J^a_{\mu\nu}$ are essentially 4d tensors in the horizontal space

Fierz

$-\frac{1}{2}\epsilon_{\mu\nu}{}^{\lambda\rho\sigma} R_\lambda J^a_{\rho\sigma} = 0 \;\Rightarrow\;$ $J^a_{\mu\nu}$ are anti-self-dual in the horizontal space

$J^a_{\mu\nu}$ enjoy the quarternion algebra

Fierz $\rightarrow$ $J^a_{\hat{m}\hat{k}} J^b_{\hat{k}\hat{n}} = \delta_{ab}\delta_{\hat{m}\hat{n}} + i\epsilon_{abc} J^c_{\hat{m}\hat{n}}$

$\gamma_\mu \xi R^\mu = \xi S \;\Rightarrow\; \gamma_{\hat{5}} \xi = \xi$

ξ has positive chirality

γ-matrices

$$\gamma^{\hat{1},\hat{2},\hat{3}} = \begin{pmatrix} & -i\tau_{1,2,3} \\ i\tau_{1,2,3} & \end{pmatrix} \quad \gamma^{\hat{4}} = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} \quad \gamma^{\hat{5}} = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}$$

With this convention we can take a gauge with

$$\xi \propto \overbrace{\begin{pmatrix} \mathbb{1}_2 \\ 0 \end{pmatrix}}^{2} \Big\updownarrow 4$$

by $Sp(2)_r \times Sp(1)_R$ rotation.

$(\xi \in (4, 2))$

We can span the spinor space by the basis

$$\underbrace{\gamma^{\hat{5}}\xi = \xi \propto \begin{pmatrix} \mathbb{1}_2 \\ 0 \end{pmatrix},\ \gamma^{\hat{1},\hat{2},\hat{3}}\xi \propto \begin{pmatrix} 0 \\ i\tau_{1,2,3} \end{pmatrix},\ \gamma^{\hat{4}}\xi \propto \begin{pmatrix} 0 \\ \mathbb{1}_2 \end{pmatrix}}_{\substack{\gamma^{\hat{\mu}}\xi \\ 5}},\ \underbrace{\xi\tau_a = \begin{pmatrix} \tau_a \\ 0 \end{pmatrix}}_{\substack{\xi\tau_a \\ 3}}$$

Killing spinor eq

Pan (1308.1567)
Y.I-Matsuno (1404.0210)

$$\delta_Q(\xi)\Psi_\mu = D_\mu \xi - f_{\mu\nu}\gamma^\nu \xi + \frac{1}{4}\gamma_{\mu\rho\sigma}v^{\rho\sigma}\xi - t\gamma_\mu \xi = 0$$

By using the spinor basis we deompose this into

$$\xi \gamma_{\hat{\mu}} \delta_Q(\xi)\Psi_\nu = 0 \quad - (1)$$

$$\xi T_a \delta_Q(\xi)\Psi_\mu = 0 \quad - (2)$$

①

$$\boxed{\begin{aligned} 0 &= (\xi \gamma_{\hat\lambda}\, \delta_Q \psi_{\hat\mu}) \\ &= \frac{1}{2} D_{\hat\mu} R_{\hat\lambda} - S f_{\hat\mu\hat\lambda} - \frac{S}{4} \epsilon_{\hat 5\hat\mu\hat\lambda\hat\rho\hat\sigma} v^{\hat\rho\hat\sigma} + S t_a J^a_{\hat\mu\hat\lambda} \end{aligned}}$$

Symmetric part

$$D_\mu R_\lambda + D_\lambda R_\mu = 0$$

$\Rightarrow$ R^μ is a Killing vector

The background vielbein

$$e^{\hat m} = e_n^{\hat m}\, dx^n, \quad e^{\hat 5} = S\,(dx^5 + V_m dx^m)$$

$e_n^{\hat m}$, S, V_m depend on x^m only.

independent of x^5

(1)

$$\boxed{\begin{aligned} 0 &= (\xi \gamma_{\hat\lambda} \delta_Q \psi_{\hat\mu}) \\ &= \frac{1}{2} D_{\hat\mu} R_{\hat\lambda} - S f_{\hat\mu\hat\lambda} - \frac{S}{4} \epsilon_{\hat 5 \hat\mu \hat\lambda \hat\rho \hat\sigma} v^{\hat\rho\hat\sigma} + S t_a J^a_{\hat\mu\hat\lambda} \end{aligned}}$$

$(\hat\lambda, \hat\mu) = (5, m)$ components

$$f_{m5} = \frac{1}{2} \partial_m S \qquad \left(\begin{array}{l} f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \\ U(1)_Z \text{ field strength} \end{array} \right)$$

This can be locally solved by

$$a = a_m dx^m + \frac{1}{2} S\, dx^5 \qquad (\text{up to } U(1)_Z \text{ gauge tr.})$$

with $a_m(x^m)$ independent of x^5

①

$$\boxed{\begin{aligned} 0 &= (\xi \gamma_{\hat\lambda} \delta_Q \psi_{\hat\mu}) \\ &= \frac{1}{2} D_{\hat\mu} R_{\hat\lambda} - S f_{\hat\mu\hat\lambda} - \frac{S}{4} \epsilon_{\hat 5 \hat\mu \hat\lambda \hat\rho \hat\sigma} v^{\hat\rho\hat\sigma} + S t_a J^a_{\hat\mu\hat\lambda} \end{aligned}}$$

anti-sym part of the horizontal part $(\hat\mu, \hat\lambda) = (\hat p, \hat q)$ can be solved by

$$v^{\hat p \hat q} = \epsilon_{\hat p \hat q \hat m \hat n} \left(\frac{S}{4} V_{\hat m \hat n} - f_{\hat m \hat n} + t_a J^a_{\hat m \hat n} \right)$$

$$(V_{mn} = \partial_m V_n - \partial_n V_m)$$

2

$$\boxed{\bar{\xi}\, T_a\, \delta_Q(\xi)\psi_\mu = 0}$$

$$\delta_Q(\xi)\psi_\mu = D_\mu \xi + \cdots$$

$$= \partial_\mu \xi + \frac{1}{4}\omega_{\mu\hat{m}\hat{n}}\gamma^{\hat{m}\hat{n}}\xi - V_\mu^a T_a \xi + \cdots$$

We can use ② to give the $SU(2)_R$ gauge field

$$V_{\hat{5}}^a = \frac{1}{4}\omega_{\hat{5}\hat{p}\hat{q}} J_{\hat{p}\hat{q}}^a + \frac{1}{4} J_{\hat{p}\hat{q}}^a v^{\hat{p}\hat{q}} - t_a$$

$$V_{\hat{m}}^a = \frac{1}{4}\omega_{\hat{m}\hat{p}\hat{q}} J_{\hat{p}\hat{q}}^a + \frac{1}{2} J_{\hat{m}\hat{p}}^a v^{\hat{p}\hat{5}}$$

Solution to the Killing spinor eq. $\delta\psi_\mu = 0$.

vielbein	$e_\mu^{\hat{\nu}}$	$e_n^{\hat{m}}$, V_m, S
$U(1)_Z$ gauge field	a_μ	a_m, a_5
anti-sym tensor	$v^{\mu\nu}$	$v^{\hat{m}\hat{n}}$, $v^{\hat{m}\hat{5}}$
$Sp(1)_R$ triplet	t_a	t_a
$Sp(1)_R$ gauge field	V_μ^a	V_m^a, V_5^a
singlet scalar	C	C

red : written in terms of other fields

green : independent of x^5

blue : unfixed

$\boxed{\text{Auxiliary eq.}}$ Y. I-Matsuno (1404.0210)

$$0 = \delta_Q(\xi)\eta = -2\gamma_\nu \xi D_\mu v^{\mu\nu} + \xi C + 4(\not{D} t)\xi$$
$$+ 8(\not{f} - \not{v}) t \xi + \gamma^{\mu\nu\rho\sigma} \xi f_{\mu\nu} f_{\rho\sigma}$$

With the spinor basis we decompose this into

$$0 = S^{-1}(\xi \delta_Q \eta) \quad - ③$$
$$0 = S^{-1}(\xi \gamma_{\hat{m}} \delta_Q \eta) \quad - ④$$
$$0 = S^{-1}(\xi T_a \delta_Q \eta) \quad - ⑤$$

3

$$0 = S^{-1}(\xi \delta_Q \eta)$$
$$= -2D_\mu v^{\mu\hat{5}} + C + 4t_a J^a_{\hat{m}\hat{n}}(f^{\hat{m}\hat{n}} - v^{\hat{m}\hat{n}}) + \epsilon^{(4)}_{\hat{m}\hat{n}\hat{p}\hat{q}} f^{\hat{m}\hat{n}} f^{\hat{p}\hat{q}}$$

This can be used to determine C

$$C = 2D_\mu v^{\mu\hat{5}} - 4t_a J^a_{\hat{m}\hat{n}}(f^{\hat{m}\hat{n}} - v^{\hat{m}\hat{n}}) - \epsilon^{(4)}_{\hat{m}\hat{n}\hat{p}\hat{q}} f^{\hat{m}\hat{n}} f^{\hat{p}\hat{q}}$$

By using relations we have obtained the remaining equations ④ and ⑤ are drastically simplified.

④ $$0 = S^{-1}(\bar{\xi}\gamma_{\hat{m}}\delta_Q\eta) = 2S^{-1}\partial_5 v_{\hat{m}\hat{5}}$$

⑤ $$0 = S^{-1}(\bar{\xi}\tau_a\delta_Q\eta) = 4S^{-1}\partial_5 t_a$$

$\Longrightarrow$ $v_{\hat{m}\hat{5}}$ and t_a are independent of x^5

We have completely solved $\delta_Q\psi_\mu = \delta_Q\eta = 0$.

The solution of $\delta_Q \psi_\mu = \delta_Q \eta = 0$

vielbein	$e_\mu^{\hat{\nu}}$	$e_n^{\hat{m}}$, V_m, S
$U(1)_Z$ gauge field	a_μ	a_m, a_5
anti-sym tensor	$v^{\mu\nu}$	$v^{\hat{m}\hat{n}}$, $v^{\hat{m}\hat{5}}$
$Sp(1)_R$ triplet	t_a	t_a
$Sp(1)_R$ gauge field	V_μ^a	V_m^a, V_5^a
singlet scalar	C	C

red : written in terms of other fields

green : independent of x^5

All fields are independent of x^5.

The general solution

$$\xi_\alpha{}^I = \sqrt{\frac{S}{2}}\begin{pmatrix} 1_2 \\ 0 \end{pmatrix}$$

$$e_n{}^{\hat m} = (\text{indep})$$

$$V_m = (\text{indep})$$

$$e_5{}^{\hat 5} = S = (\text{indep})$$

$$a_{\hat m} = (\text{indep})$$

$$a_5 = \frac{1}{2} S$$

$$v_{\hat p\hat q} = \epsilon^{(4)}_{\hat p\hat q\hat m\hat n}\left(\frac{S}{4} V_{\hat m\hat n} - f_{\hat m\hat n} + t_a J^a_{\hat m\hat n}\right)$$

$$v_{\hat n\hat 5} = (\text{indep})$$

$$t_a = (\text{indep})$$

$$V^a_{\hat m} = \frac{1}{4}\omega^{(4)}_{\hat m\hat p\hat q} J^a_{\hat p\hat q} + \frac{1}{2} J^a_{\hat m\hat p} v^{\hat p\hat 5}$$

$$V^a_{\hat 5} = \frac{1}{2} J^a_{\hat m\hat n}\left(f_{\hat m\hat n} - \frac{S}{2} V_{\hat m\hat n}\right) + t_a$$

$$C = 2 D^{(4)}_{\hat m} v^{\hat m\hat 5} + 4 t_a J^a_{\hat m\hat n} f_{\hat m\hat n} + 32 t_a t_a - \epsilon^{(4)}_{\hat m\hat n\hat p\hat q}\left(f^{\hat m\hat n} - \frac{S}{2} V^{\hat m\hat n}\right)\left(f^{\hat p\hat q} - \frac{S}{2} V^{\hat p\hat q}\right)$$

The isometry

The existence of the isometry is expected from the commutation relation

$$\{\delta_Q(\xi_1), \delta_Q(\xi_2)\} = \underline{\underline{\delta_{diff}}} + \delta_{U(1)_Z} + \delta_{SU(2)_R} + \delta_{SO(5)_r}$$

The parameter for δ_{diff} is $\sim \xi_1 \gamma^\mu \xi_2$

Existence of rigid SUSY $\delta_Q(\xi)$

$\rightarrow$ Existence of isometry $\delta_{diff}(R^\mu)$

In the gauge that we have chosen,

$$(\delta_Q(\xi))^2 = \partial_5$$

Remark

We have not considered global issues.

A single coordinate patch has been focused on.

$\Downarrow$

future work

Deformation dependence of Z

Let S_0 be a theory of matter fields defined on a susy background.

(example) $S_0 = \int d^5x \sqrt{g} \left(\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \cdots \right)$

A small deformation of the background

$$S_0 \to S_0 + S_1$$

$$S_1 = \int d^5x \sqrt{g} \left[-\Delta e_\mu^{\hat{\nu}} T_{\hat{\nu}}^{\mu} + \Delta \psi_\mu S^\mu + \cdots \right]$$

If S_1 is $\delta_Q(\xi)$ exact, this deformation does not affect the partition function Z.

Question

To what extent can we realize susy preserving deformations by adding Q-exact terms to the action?

We follow the following steps.

① determine δ_Q-transf. of the fermionic currents S^μ and χ. in the curved background

② compute general δ_Q-exact action of the form

$$S_{exact} = \delta_Q \int d^5x \sqrt{g} \left(F_\mu^{I\alpha} S^\mu_{I\alpha} + F^{I\alpha} \chi_{I\alpha} \right)$$

③ compare this with general susy preserving deformation.

① Determination of $\delta_Q S^\mu$ and $\delta_Q \chi$

We assume the background is δ_Q invariant

$$\Downarrow$$

A small deformation $(\Delta e_\mu^{\hat{\nu}}, \Delta \psi_\mu, \cdots)$ are linearly transformed. ($\delta_Q e_\mu^{\hat{\nu}}$ etc. are in the literature)

$$\Downarrow$$

We can determine δ_Q(currents) so that S_1 is susy invariant.

Example

Focus on the terms in $\delta_Q S_1$ containing $\Delta\eta \equiv \eta$

$$\delta_Q S_1 = \delta_Q \int \cdots + \underbrace{\Delta V_\mu^a}_{\downarrow \delta_Q} R_a^\mu + \underbrace{\Delta \eta}_{\downarrow \delta_Q} \chi + \cdots$$

$$= \int \cdots - \frac{1}{4}(\xi \tau_a \gamma_\mu \eta) R_a^\mu + \eta \delta_Q \chi + \cdots$$

For this to vanish

$$\eta \delta_Q \chi - \frac{1}{4}(\eta \tau_a \gamma_\mu \xi) R_a^\mu + \cdots = 0$$

$$\Rightarrow \quad \delta_Q \chi = \frac{1}{4} \tau_a \gamma_\mu \xi R_a^\mu + \frac{1}{4} \tau_a \xi X_a - \frac{1}{2} \gamma_{\mu\nu} \xi M^{\mu\nu}$$
$$+ \gamma^\mu \xi D_\mu \Phi + f_{\mu\nu} \gamma^{\mu\nu} \xi \Phi + 16 \tau \xi \Phi .$$

② δ_Q exact terms

$S_{exact} = \delta_Q \int \sqrt{g}\, d^5x\, (F\chi)$ $\quad F$: x^5 indep. spinor func. (bosonic)

expansion of F by the spinor basis

$$F \to f\xi,\ \ f^a \tau_a \xi,\ \ f^{\hat{m}} \gamma_{\hat{m}} \xi$$

$$(F\chi) = f(\xi\chi) + \underline{\frac{4}{5} f^a (\xi \tau_a \chi)} - \frac{2}{5} f^{\hat{m}} (\xi \gamma_{\hat{m}} \chi)$$

The second term gives

$$S_{exact} = \delta_Q \int d^5x \sqrt{g}\, \frac{4}{5} f^a (\xi \tau_a \chi)$$

(continued)

$$= \int d^5x \sqrt{g} \Big(\underbrace{f^a R^{\hat{5}}_a}_{\Delta V^a_{\hat{5}} R^{\hat{5}}_a} + \underbrace{f^a X_a}_{\Delta \tau^a X_a} - 2 f^a J^a_{\hat{m}\hat{n}} M^{\hat{m}\hat{n}} + 4 f^a J^a_{\hat{m}\hat{n}} f^{\hat{m}\hat{n}} \Phi + 64 f^a \tau_a \Phi \Big)$$

$\cdots$

③ Comparison to $S_1 = \int \sqrt{g} \left(- \Delta e^{\hat{\nu}}_{\mu} T^{\mu}_{\hat{\nu}} + \cdots \right)$

We can read off the corresponding background deformation

$$\begin{cases} \Delta V^a_{\hat{5}} = f^a \\ \Delta \tau^a = f^a \\ \Delta v^{\hat{m}\hat{n}} = -2 f^a J^a_{\hat{m}\hat{n}} \\ \Delta C = 4 f^a J^a_{\hat{m}\hat{n}} f^{\hat{m}\hat{n}} + 64 f^a \tau_a \\ \Delta e^{\hat{\nu}}_{\mu} = \Delta a_{\mu} = \Delta v^{\hat{m}\hat{5}} = \Delta V^a_{\hat{m}} = 0 \end{cases}$$

The general solution

$$\xi_\alpha{}^I = \sqrt{\frac{S}{2}}\begin{pmatrix} 1_2 \\ 0 \end{pmatrix}$$

$$e_n{}^{\hat m} = (\text{indep})$$

$$V_m = (\text{indep})$$

$$e_5{}^{\hat 5} = S = (\text{indep})$$

$$a_{\hat m} = (\text{indep})$$

$$a_5 = \frac{1}{2} S$$

$$v_{\hat p\hat q} = \epsilon^{(4)}_{\hat p\hat q\hat m\hat n}\left(\frac{S}{4} V_{\hat m\hat n} - f_{\hat m\hat n} + t_a J^a_{\hat m\hat n}\right)$$

$$v_{\hat m\hat 5} = (\text{indep})$$

$$t_a = (\text{indep})$$

$$V^a_{\hat m} = \frac{1}{4}\omega^{(4)}_{\hat m\hat p\hat q} J^a_{\hat p\hat q} + \frac{1}{2} J^a_{\hat m\hat p} v^{\hat p\hat 5}$$

$$V^a_{\hat 5} = \frac{1}{2} J^a_{\hat m\hat n}\left(f_{\hat m\hat n} - \frac{S}{2} V_{\hat m\hat n}\right) + T_a$$

$$C = 2 D^{(4)}_{\hat m} v^{\hat m\hat 5} + 4 t_a J^a_{\hat m\hat n} f_{\hat m\hat n} + 32 t_a T_a - \epsilon^{(4)}_{\hat m\hat n\hat p\hat q}\left(f^{\hat m\hat n} - \frac{S}{2} V^{\hat m\hat n}\right)\left(f^{\hat p\hat q} - \frac{S}{2} V^{\hat p\hat q}\right)$$

This is generated by the change of the independent field t^a.

$$\Delta t^a = f^a$$

We can freely change t^a by Q-exact def.

t^a does not affect Z (at least locally).

Other terms in $\delta_Q \int F X$ are also computed similarly

Q exact terms in $\delta_Q \int F_\mu S^\mu$ are more complicated.

Let us find a shortcut.

We followed the following steps.

① consider the small deformation

$$S_1 = \int \Delta A_i^B J_i^B + \Delta A_i^F J_i^F$$

where (A_i^B, A_i^F) ⋯ gravity multiplet

(J_i^B, J_i^F) ⋯ current multiplet

② Susy transf. of ΔA_i^B are linear in $\Delta A_i^F \equiv A_i^F$

$$\delta_Q \Delta A_i^B = \Delta A_j^F M_{ji}$$

From the SUSY invariance of S_1

$$0 = \delta_Q S_1 = \int \underbrace{\delta_Q \Delta A_i^B J_i^B}_{= \Delta A_i^F M_{ij} J_j^B} - \Delta A_i^F \delta_Q J_i^F$$

we obtain

$$\delta_Q J_i^F = M_{ij} J_i^B$$

③ general Q-exact deformation

$$\delta_Q \int f_i J_i^F = \int f_i \delta_Q J_i^F = \int f_i M_{ij} J_j^B$$

By comparing this to $S_1 = \int \Delta A_i J_i^B$

we can read off

$$\Delta A_i^B (Q\text{-exact}) = f_i M_{ij}$$
$$= \delta_Q \Delta A_i^B \Big|_{A_i^F \to f_i}$$

Shortcut

Q-exact def. can be obtaind

from susy trans. by the replacement

$$A_i^F \to f_i$$

fermions parameters.

Susy transf of bosonic components in Weyl.

$$\delta_Q e_\mu^{\hat{\nu}} = -2(\xi\gamma^{\hat{\nu}}\psi_\mu)$$

$$\delta_Q a_\mu = -(\xi\psi_\mu)$$

$$\delta_Q V_\mu^a = -\frac{1}{4}(\xi\tau_a\gamma_\mu\eta) + (\xi\tau_a\gamma^\lambda R_{\lambda\mu}(Q)) + (\xi\tau_a\gamma^{\rho\sigma}f_{\rho\sigma}\psi_\mu)$$
$$-(\xi\tau_a\gamma^{\rho\sigma}v_{\rho\sigma}\psi_\mu) + 6(\xi\psi_\mu)t_a$$

$$\delta_Q t_a = -\frac{1}{4}(\xi\tau_a\eta),$$

$$\delta_Q v_{\mu\nu} = \frac{1}{2}(\xi\gamma_{\mu\nu\rho\sigma}R^{\rho\sigma}(Q)) + \frac{1}{2}(\xi\gamma_{\mu\nu}\eta)$$

$$\delta_Q C = -(\xi\hat{\not{D}}\eta) - 11(\xi t\eta) - \frac{3}{4}(\xi\gamma_{\mu\nu}v^{\mu\nu}\eta) - 4(\xi t\gamma^{\mu\nu}R_{\mu\nu}(Q))$$

By the replacement $(\eta, \psi_\mu) \to (-\frac{5}{4}f_a\tau_a\xi, 0)$

we reproduce the background deformation corresponding to

$$S_{exact} = \delta_Q \int d^5x\sqrt{g}\,\frac{4}{5}f^a(\xi\tau_a\chi)$$

With this prescription, we can realize all the susy-pres, deformations as Q-exact deformations.

Comments

- We have not taken care of global issues.

- We can only claim that "local" degrees of freedom do not affect Z.

- In some examples (S^5, $S^4 \times S^1$, etc.) Z depends on a few parameters.

- Similar analysis can be done for background vector multiplets.

$\boxed{\text{Summary}}$

- In the framework of 5d $\mathcal{N}=1$ supergravity we constructed susy-pres. background.

- We took Euclidean signature, and imposed symplectic Majorana cond. on ξ just for simplicity.

- As long as we focus on a single coordinate patch susy-prs. deformations can be realized as Q-exact deformatios.

Future directions.

- Removing the restriction on ξ
  - Complex ξ,
  - Lorenzian signature
- Global issues
  - How many parameters can affect Z for a given topology?
- reduction to 4d
  - With the isometry we can reduce the sol. to 4d. (straightforward?)

- Other off-shell SUGRA.

$$\text{5d conformal SUGRA} \xrightarrow{\text{gauge fixing}} \begin{cases} \text{SUGRA we used here} \\ \text{another SUGRA} \\ \cdots \end{cases}$$

- Two 5d solutions from the same 6d sol.

$$\begin{array}{ccc} & (2,0) & \\ \swarrow & & \searrow \\ \text{5d SYM on } \mathcal{M}_1 & & \text{5d SYM on } \mathcal{M}_2 \end{array}$$

It is easy to construct a pair $(\mathcal{M}_1, \mathcal{M}_2)$

$$Z(\mathcal{M}_1) \stackrel{?}{=} Z(\mathcal{M}_2)$$

Thank You !