SELF-DUAL STRINGS AND 2D SYM

Kazuo Hosomichi
(YITP, Kyoto)

Joint work with Sungjay Lee (U of Chicago)

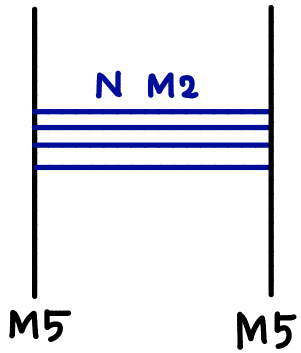
arXiv:1406.1802

Introduction

It is not fully understood what 2D field theory describes the collective dynamics of self-dual strings

(fundamental DOF on M5-brane worldvolume)

When M5-branes are separated, they are described as M2-branes suspended between M5-branes.



"M-strings"

M-strings [Haghighat-Iqbal-Kozcaz-Lockhart-Vafa]

- T^2-partition function of multi-M-strings proposed.

  … extracted from multiple M5 amplitudes

  (refined topological vertex.)

- 2D worldsheet theory proposed

  … based on D-brane construction in several dual frameworks

Our goal : Re-derive these from ABJM.

Our ABJM analysis was able to reproduce the known results on M-strings (elliptic genus, ⋯) only partially.

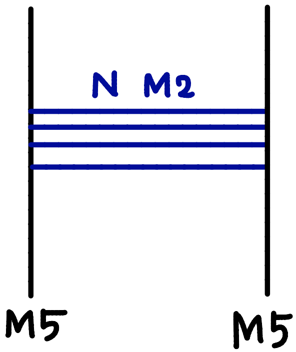
Setup

$\natural = 10$

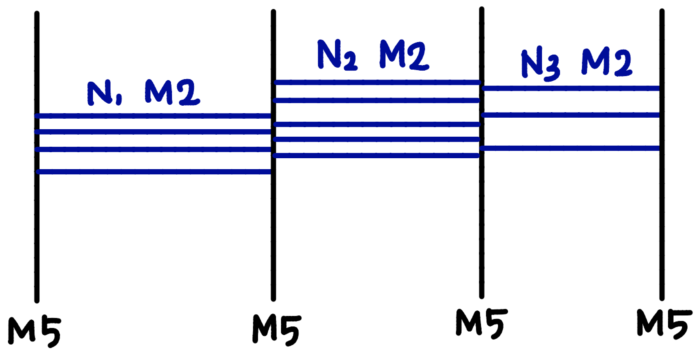
	0	1	2	3	4	5	6	7	8	9	$\natural$
M2-brane	—	—	⊣	·	·	·	·	·	·	·	·
M5-brane	—	—	·	—	—	—	—	·	·	·	·

Directions 3–$\natural$: $\mathbb{C}^4/\mathbb{Z}_k$

[simplest]



[general]



SUSY

M2-branes : $\Gamma^{012} Q = Q$

M5-branes : $\Gamma^{013456} Q = Q$

$\mathbb{Z}_k$-orbifold : $\exp \frac{\pi}{k} \left(\Gamma^{34} + \Gamma^{56} + \Gamma^{78} + \Gamma^{9\natural} \right) Q = Q$

Unbroken SUSY

eigenspinor of $(\Gamma^{01}, \Gamma^{2}, i\Gamma^{34}, i\Gamma^{56}, i\Gamma^{78}, i\Gamma^{9\natural})$

$(++++-+-)$	$(--++++)$
$(+++--+)$	$(--++--)$
$(++-++-)$	$(----++)$
$(++-+-+)$	$(------)$

projected out by $\mathbb{Z}_k$ $(k \geq 3)$: $(--++++)$, $(------)$

2D $\mathcal{N} = (4,2)$ SUSY

R-symmetry

	0	1	2	3	4	5	6	7	8	9	♮
M2-brane	—	—	⊣	·	·	·	·	·	·	·	·
M5-brane	—	—	·	—	—	—	—	·	·	·	·

$$SO(8) \to SO(4)_{3456} \times SO(4)_{789\natural}$$

$$\simeq \underline{SU(2)_1} \times \underline{SU(2)_2} \times \underline{SU(2)_3} \times \underline{SU(2)_4}$$

2D $\mathcal{N}=(4,4)$ SUSY : $(\underline{2},\underline{1},\underline{2},\underline{1})_+ \oplus (\underline{1},\underline{2},\underline{1},\underline{2})_-$

R-symmetry

	0	1	2	3	4	5	6	7	8	9	♮
M2-brane	—	—	⊣	·	·	·	·	·	·	·	·
M5-brane	—	—	·	—	—	—	—	·	·	·	·

$$SO(8) \to SO(4)_{3456} \times SO(4)_{789\natural}$$

$$\simeq \underline{SU(2)_1} \times \underline{SU(2)_2} \times \underline{SU(2)_3} \times \underline{SU(2)_4}$$

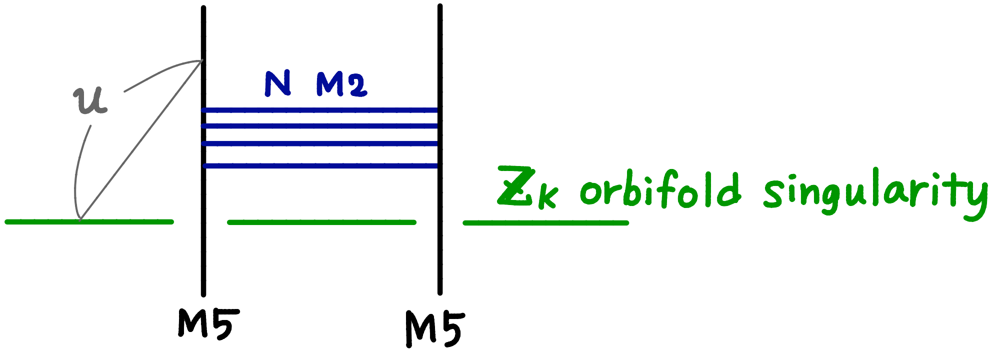
2D $\mathcal{N}=(4,4)$ SUSY : $(\underline{2},\underline{1},\underline{2},\underline{1})_{+} \oplus (\underline{1},\underline{2},\underline{1},\underline{2})_{-}$

[with $\mathbb{Z}_k$ orbifolding]

$$SU(4) \to SU(2)_1 \times SU(2)_3 \times U(1)$$

or $SO(6) \to SO(4) \times SO(2)$

2D $\mathcal{N}=(4,2)$ SUSY : $(2,2;0)_{+} \oplus (0,0;\pm 1)_{-}$



R-symmetry becomes even smaller if $u > 0$.

ABJM model

… $U(N)_k \times U(N)_{-k}$ CS theory

with bi-fund scalars Z_a, spinors Ψ^a $(a = 1, 2, 3, 4)$

$$\mathcal{L} = k \cdot \mathcal{L}_{CS}[A] - k \cdot \mathcal{L}_{CS}[\tilde{A}]$$
$$+ \mathrm{Tr}\left[- D_m \bar{Z}^a D^m Z_a + i \bar{\Psi}_a \gamma^m D_m \Psi^a \right]$$
$$+ \mathcal{O}(Z^2 \Psi^2) + \mathcal{O}(Z^6)$$

- $\mathcal{N} = 6$ SUSY : ξ_{ab} ; $(\xi_{ab})^\dagger = \frac{1}{2} \epsilon^{abcd} \xi_{cd}$

Boundary condition [Berman-Perry-Sezgin-Thompson]

[B.C. on scalars]

	0	1	2	3	4	5	6	7	8	9	♮
M2-brane	—	—	—|	·	·	·	·	·	·	·	·
M5-brane	—	—	·	—	—	—	—	·	·	·	·

Directions 3–6: Z_I $(I=1,2)$ … "Nahm"

Directions 7–♮: Z_A $(A=3,4)$ … Dirichlet

[B.C. on supercurrent J^m_{ab}]

$$J^2_{IA+} = J^2_{IJ-} = J^2_{AB-} = 0$$

$\Rightarrow$ Q_{IA+}, Q_{IJ-}, Q_{AB-} are preserved.

Boundary condition

- spinors: $\Psi^{I}_{+} = \Psi^{A}_{-} = 0$

- scalars: [Dirichlet b.c.]

$$Z_A = \text{const.}$$

[Nahm b.c.]

$$D_y Z_I = \frac{2\pi}{k}\left(Z_I \bar{Z}^J Z_J - Z_J \bar{Z}^J Z_I\right) \quad (y = x_2)$$

— — —

$$Z_A \bar{Z}^I Z_B = Z_B \bar{Z}^I Z_A\,, \quad Z_A \bar{Z}^B Z_I = Z_I \bar{Z}^B Z_A$$

Boundary condition

- spinors : $\Psi_+^I = \Psi_-^A = 0$
- scalars : $Z_A = \text{const.}$ (Dirichlet)

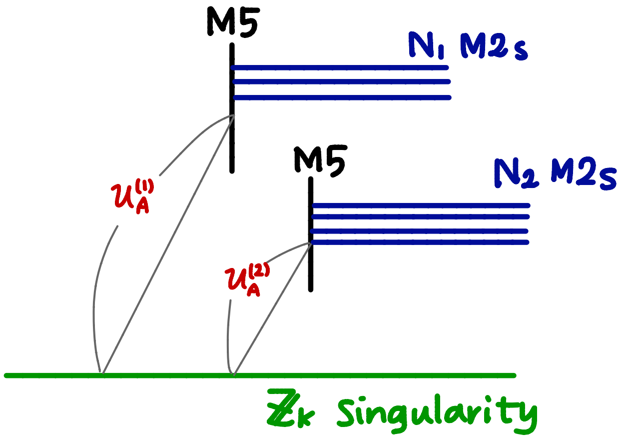
$$D_y Z_I = \frac{2\pi}{k}\left(Z_I \bar{Z}^J Z_J - Z_J \bar{Z}^J Z_I\right) \quad \text{(Nahm)}$$

$$Z_A \bar{Z}^I Z_B = Z_B \bar{Z}^I Z_A, \quad Z_A \bar{Z}^B Z_I = Z_I \bar{Z}^B Z_A$$

[example]

$$Z_A = \begin{pmatrix} u_A^{(1)} \cdot \mathbb{1}_{N_1 \times N_1} & & \\ & u_A^{(2)} \cdot \mathbb{1}_{N_2 \times N_2} & \\ & & \ddots \end{pmatrix}$$

$$Z_I = \begin{pmatrix} N_1 \times N_1 & & \\ & N_2 \times N_2 & \\ & & \ddots \end{pmatrix}$$



Boundary condition on gauge fields

[Dirichlet b.c. on Z_A]

$$D_\mu Z_A = 0 \quad (\mu = 0, 1)$$

$$\Rightarrow \quad 0 = [D_\mu, D_\nu] Z_A = -i(F_{\mu\nu} Z_A - Z_A \tilde{F}_{\mu\nu})$$

$\therefore$ A_μ and $\tilde{A}_\mu$ are unitary equivalent.

[example]

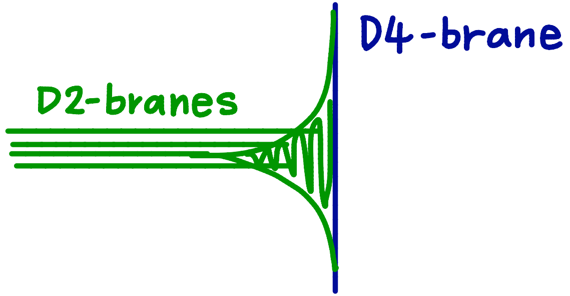
If $Z_A = u_A \cdot \mathbb{1}_{N\times N}$, then $A_\mu = \tilde{A}_\mu$ (or $u_A = 0$)

Nahm boundary condition (or Nahm pole)

... singular behavior of Nahm configurations at the boundary.

$$\frac{d}{dy}X_i + \epsilon_{ijk} X_j X_k = 0 \qquad X_i \sim \frac{T_i}{y} \quad (T_i : SU(2) \text{ generator})$$

D2 ⊣ D4 system



Let's revisit its M-theory uplift.

Nahm configurations in ABJM

½ BPS configurations

(preserving 2D $\mathcal{N}=(4,2)$ SUSY) satisfy

$$D_y Z_I = \frac{2\pi}{k}\left(Z_I \bar{Z}^J Z_J - Z_J \bar{Z}^J Z_I\right)$$

$$D_y Z_A = \frac{2\pi}{k}\left(Z_A \bar{Z}^B Z_B - Z_B \bar{Z}^B Z_A\right)$$

$$Z_I \bar{Z}^A Z_J = Z_J \bar{Z}^A Z_I\,, \quad Z_I \bar{Z}^J Z_A = Z_A \bar{Z}^J Z_I$$

$$Z_A \bar{Z}^I Z_B = Z_B \bar{Z}^I Z_A\,, \quad Z_A \bar{Z}^B Z_I = Z_I \bar{Z}^B Z_A$$

$$D_y Z_I = \frac{2\pi}{k}\left(Z_I \bar{Z}^J Z_J - Z_J \bar{Z}^J Z_I\right) \quad (1)$$

$$D_y Z_A = \frac{2\pi}{k}\left(Z_A \bar{Z}^B Z_B - Z_B \bar{Z}^B Z_A\right) \quad (2)$$

$$\left.\begin{array}{l} Z_I \bar{Z}^A Z_J = Z_J \bar{Z}^A Z_I, \quad Z_I \bar{Z}^J Z_A = Z_A \bar{Z}^J Z_I \\ Z_A \bar{Z}^I Z_B = Z_B \bar{Z}^I Z_A, \quad Z_A \bar{Z}^B Z_I = Z_I \bar{Z}^B Z_A \end{array}\right\} (3)$$

(3) $\Longrightarrow$ if $Z_I \neq 0$ and $Z_A \neq 0$, they are all diagonal, so y-independent.

(1) shows Nahm pole behavior only when $Z_A = 0$.

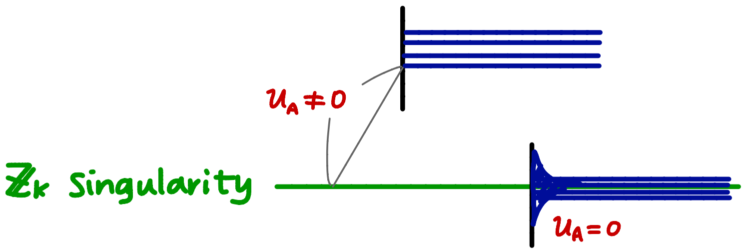
M5-branes are "D4-like" only when $u_A = 0$.

They are NS5-like when $u_A \neq 0$.

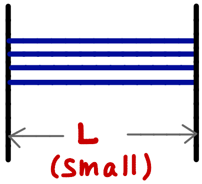
M5-branes in ABJM (Summary)

- $U(N) \times U(N) \to U(N)$ $\quad A_\mu = \tilde{A}_\mu$
- Dirichlet b.c. $\quad Z_A = u_A \quad (A=3,4)$
- Nahm b.c. $\quad D_y Z_I = \frac{2\pi}{k}\left(Z_I \bar{Z}^J Z_J - Z_J \bar{Z}^J Z_I\right) \quad (I,J=1,2)$

Nahm pole behavior only when $u_A = 0$.



2D theory on "ABJM slab"



Let us work out the 2D $\mathcal{N}=(4,2)$ theory by dimensional reduction,

$$\mathcal{L}_{2D} = \int_0^L dy \cdot \mathcal{L}_{ABJM}(y\text{-dependence dropped})$$

$$+ \mathcal{L}_{boundary}\Big|_{y=L} - \mathcal{L}_{boundary}\Big|_{y=0} .$$

Dimensional Reduction of $\mathcal{L}_{ABJM}$

$\mathcal{L}_{ABJM}$ (y-dependence dropped)

$$= \frac{k}{2\pi} \mathrm{Tr}\left[A_y \cdot F_{01} - \tilde{A}_y \cdot \bar{F}_{01} \right]$$

$$+ \mathrm{Tr}\Big[- D_\mu \bar{Z}^A \cdot D_\mu Z_A - D_y \bar{Z}^A \cdot D_y Z_A$$

$$- D_\mu \bar{Z}^I \cdot D_\mu Z_I - \underbrace{D_y \bar{Z}^I \cdot D_y Z_I + \mathcal{O}(z^6)}_{} + (\text{fermi}) \Big]$$

Recall $\underline{F_{\mu\nu} = \bar{F}_{\mu\nu}}$, $\underline{Z_A = u_A}$ and denote $\underline{A_y - \tilde{A}_y \equiv \sigma}$.

※ $\underbrace{(\cdots)}_{}$ = $(\text{Nahm eq.})^2$ if combined with $\mathcal{L}_{boundary}$

Dimensional Reduction of $\mathcal{L}_{ABJM}$

$$\mathcal{L}_{ABJM}\,(\text{y-dependence dropped}) \quad + \mathcal{L}_{bdry}$$

$$= \frac{k}{2\pi} \mathrm{Tr}\left[A_y \cdot F_{01} - \tilde{A}_y \cdot \bar{F}_{01} \right]$$

$$+ \mathrm{Tr}\Big[- D_\mu \bar{Z}^A \cdot D_\mu Z_A - D_y \bar{Z}^A \cdot D_y Z_A$$

(Dirichlet b.c.)

$$- D_\mu \bar{Z}^I \cdot D_\mu Z_I - D_y \bar{Z}^I \cdot D_y Z_I + \mathcal{O}(z^6) + (\text{fermi}) \Big]$$

(Nahm b.c.)

$$= \mathrm{Tr}\left[\frac{k}{2\pi} \sigma \cdot F_{01} - \sigma^2 \cdot |u_A|^2 - D_\mu \bar{Z}^I D^\mu Z_I + \cdots \right]$$

2D Yang-Mills

2D theory on ABJM slab

For non-zero u_A one finds

$$\mathcal{L}_{2D} = L \cdot \mathrm{Tr}\left[\frac{k}{2\pi} \sigma \cdot F_{01} - \sigma^2 \cdot |u_A|^2 - D_\mu \bar{Z}^I D^\mu Z_I + \cdots \right]$$

$$= \quad \text{2D } \mathcal{N}=(4,4) \text{ SYM with } g^2 = \frac{16\pi^2 u^2}{k^2 L}$$

(D2 – NS5 theory)

[Note] $\sigma \equiv A_y - \tilde{A}_y$ is periodic.

$$W \equiv \mathrm{Tr}\left[P\exp i\int_0^L dy\, A_y \cdot P\exp i\int_L^0 dy\, \tilde{A}_y \right]$$

$$\simeq \mathrm{Tr} \exp iL\sigma \quad \text{is gauge-invariant.}$$

2D theory on ABJM slab

For $u_A = 0$, the Lagrangian simplifies

$$\mathcal{L}_{2D} = L \cdot \mathrm{Tr}\Big[\frac{k}{2\pi}\sigma \cdot F_{01} - D_\mu \bar{Z}^I D^\mu Z_I + i\bar{\Psi}_I \gamma^\mu D_\mu \Psi^I + i\bar{\Psi}_A \gamma^\mu D_\mu \Psi^A \Big]$$

(anti-chiral) (chiral)

- has $\mathcal{N}=(4,4)$ SUSY and $SU(2)^4$ R-symmetry
- No scalar potential!

  Can't reproduce the moduli space $\mathrm{Sym}^{\otimes N}(\mathbb{C}^2/\mathbb{Z}_k)$

A natural question :

How can the scalar σ be periodic in

$$\mathcal{L}_{2D} = L \cdot \mathrm{Tr}\left[\frac{k}{2\pi} \sigma \cdot F_{01} + \sigma^2 u^2 + (\text{matter}) \right] \ ?$$

The "1st order" YM Lagrangian with periodic σ

was proposed before by [Aganagic-Ooguri-Saulina-Vafa].

... "q-deformed" 2D YM

q-deformed 2DYM

[AOSV] studied topologial string on a CY :

$$L_1 \oplus L_2 \to \Sigma_h . \qquad (\deg L_1, \deg L_2) = (2h+2-p,\ p)$$

- Lagrangian for N D4-branes wrapping $L_2 \to \Sigma_h$

$$S_{D4} = \frac{1}{g_s} \int_{L_2 \to \Sigma_h} \mathrm{Tr}[F \wedge F]$$

$$= k \int_{M_{h,p}} \mathcal{L}_{CS}[A] \qquad (k \sim g_s^{-1})$$

$M_{h,p}$ = S^1-bundle over Σ_h of degree p.

q-deformed 2DYM

A naive dim. reduction of $\boxed{k\int_{M_{h,p}} \mathcal{L}_{CS}[A]}$ along S^1

$\Rightarrow$ (1st order) 2D YM with a periodic scalar σ

and $g_{YM}^2 = ikp$. Imaginary & quantized.

However, CS theory is topological so does not dimensionally reduce.

Observables get q-deformed due to KK-modes.

Example: partition function

2D YM on Σ_h:

$$Z = \text{const}\cdot\sum_{\lambda:\text{reps}} d(\lambda)^{2-2h}\exp\left[-\frac{1}{2}g^2C_2(\lambda)\right]$$

$$d(\lambda) = \prod_{\alpha>0}\frac{\alpha\cdot(\lambda+\rho)}{\alpha\cdot\rho}$$

3D CS on $M_{h,p}$:

$$Z = \text{const}\cdot\sum_{\lambda:\text{reps}} d_q(\lambda)^{2-2h}\exp\left[-\frac{i}{2}\hat{k}p\,C_2(\lambda)\right]$$

$$d_q(\lambda) = \prod_{\alpha>0}\frac{[\alpha\cdot(\lambda+\rho)]_q}{[\alpha\cdot\rho]_q}$$

$$[x]_q = \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}},\quad q=\exp(2\pi i/\hat{k}),\quad \hat{k}\equiv k+N$$

Back to the ABJM slab,

- $U(N) \times U(N)$ theory on interval $\simeq$ $U(N)$ on S^1
- the gauge coupling arises from VEV of matter scalars

$\Rightarrow$ needs to explain the periodicity of σ

differently from [AOSV].

Periodicity of σ in ABJM slab

$\cdots$ associated to large gauge transformations

[example : U(1)×U(1)] $\sigma' = \sigma + \frac{2\pi n}{L}$, $Z_a' = Z_a \cdot \exp\left(\frac{2\pi i n y}{L}\right)$

incompatible with dim. reduction $Z_A = u_A$

Restoring all the KK modes one finds,

$$\int dy\, D_y \bar{Z}^A D_y Z_A = L u^2 \sigma^2 + (\text{modes})$$

$$= \frac{4u^2}{L} \sin^2 \frac{\sigma L}{2} + (\text{complete square in modes})$$

Comment.

- periodicity $\sigma \sim \sigma + \frac{2\pi}{L}$ becomes unimportant in the naive limit $L \to 0$.

- It will remain important in the limit

$$L \to 0, \quad u_A \to 0 \quad \text{s.t.} \quad g^2 = \frac{16\pi^2 u^2}{k^2 L} \quad \text{fixed}.$$

ABJM slab : Summary

- Dimensional reduction of ABJM with M5-like boundary condition

$u_A \neq 0 \rightarrow$ $\mathcal{N}=(4,4)$ SYM with $SU(2)^3_R$, in 1st order form

periodicity of σ

$u_A = 0 \rightarrow$ "topological SYM", $\mathcal{N}=(4,4)$ SUSY & $SU(2)^4_R$

$\mathcal{L} = \mathrm{Tr}[\sigma F_{01} + \text{matter}]$

2D theory on self-dual strings : Another construction

[Haghighat - Iqbal · Kozcaz - Lockhart - Vafa]

※ Hereafter $k =: 1$

	0	1	2	3	4	5	6	7	8	9	♮
M2-brane	—	—	⊣	·	·	·	·	·	·	·	·
M5-brane	—	—	·	—	—	—	—	·	·	·	·

Replace $\mathbb{R}^4$(789♮) with Taub-NUT and reduce along S^1

⟹

	0	1	2	3	4	5	6	7	8	9
D2-brane	—	—	⊣	·	·	·	·	·	·	·
NS5-brane	—	—	·	—	—	—	—	·	·	·
D6-brane	—	—	—	—	—	—	—	·	·	·

"IIA-brane model"

IIA-brane model

… 2D $\mathcal{N}=(0,4)$ SUSY theory

- D2-D6 composite without NS5 gives

  2D $\mathcal{N}=(4,4)$ U(N) theory with 1adj⊕1 fund. hypers

	scalars	spinors	vectors
vector	$Y^{A\dot{A}}$	$\lambda_-^{\dot{I}A}$, $\lambda_+^{\dot{I}\dot{A}}$	A_μ
adj. hyper	$Z^{I\dot{I}}$	$\Psi_-^{I\dot{A}}$, Ψ_+^{IA}	
fund. hyper	$q^{\dot{I}}$	$\psi_-^{\dot{A}}$, ψ_+^{A}	

indices $I, \dot{I}, A, \dot{A}$ are for $SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times SU(2)_4$

- NS5 freezes some of the fields, breaks SUSY to $\mathcal{N}=(0,4)$.

IIA-brane model

… 2D $\mathcal{N}=(0,4)$ SUSY theory

- D2-D6 composite without NS5 gives

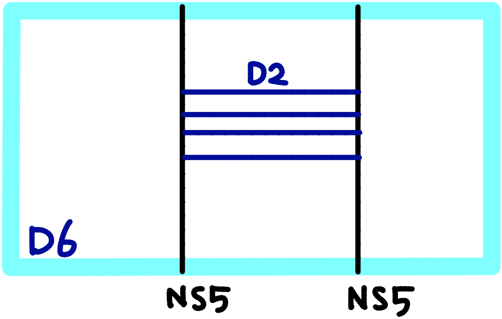
  2D $\mathcal{N}=(4,4)$ U(N) theory with 1 adj $\oplus$ 1 fund. hypers

	scalars	spinors	vectors
vector	$\langle Y^{A\dot{A}} \rangle = u^{A\dot{A}}$	$\lambda_-^{\dot{I}A}$, $\lambda_+^{\dot{I}\dot{A}}$	A_μ
adj. hyper	$Z^{I\dot{I}}$	$\Psi_-^{I\dot{A}}$, Ψ_+^{IA}	
fund. hyper	$q^{\dot{I}}$	$\psi_-^{\dot{A}}$, ψ_+^{A}	

indices $I, \dot{I}, A, \dot{A}$ are for $SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times SU(2)_4$

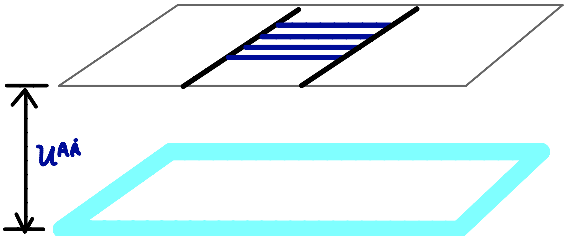
- NS5 freezes some of the fields, breaks SUSY to $\mathcal{N}=(0,4)$.

Brane configuration



- $u^{A\dot{A}} = 0$

… $\mathcal{N} = (0,4)$ SUSY & $SU(2)^4$ symmetry



- $u^{A\dot{A}} \neq 0$

… $SU(2)^3$ symmetry

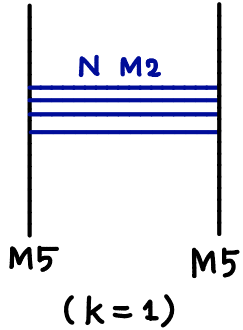
… fund. matter becomes massive

… enhanced $\mathcal{N} = (4,4)$ SUSY at $u \to \infty$

2 theories for self-dual strings (summary)

ABJM slab	IIA brane model
[u=0] topological YM + matter $\mathcal{L} = \mathrm{Tr}[\sigma F_{01} + (\text{matter})]$ (4,4) SUSY ; $SU(2)^4$	[u=0] U(N) YM with adj + fund. matter (0,4) SUSY ; $SU(2)^4$
[u≠0] (4,4) SYM ; $SU(2)^3$	[u≠0] fund. matters get massive (0,4) SUSY ; $SU(2)^3$ → approaches (4,4) SYM in the limit $u \to \infty$

Comparing the two descriptions



We studied the elliptic genus with most general twist preserving (0,2) SUSY.

- $u=0$, $SU(2)^4$ symmetry

  ⋯ twist by $(\epsilon_1-\epsilon_2)\,J_3^{(1)} + (\epsilon_2+\epsilon_2)\,J_3^{(2)} + 2m\cdot J_3^{(3)} + (\epsilon_1+\epsilon_2)\,J_3^{(4)}$

- $u\neq 0$, $SU(2)^3$ symmetry

  ⋯ needs $2m = (\pm)(\epsilon_1+\epsilon_2)$

Elliptic genus

… $\mathcal{N} \geq (0,2)$ SUSY partition function on T^2.

… General formula [Gadde-Gukov] [Benini-Eager-Hori-Tachikawa]

- saddle points are labeled by flat gauge fields

  (r complex parameters for rank-r gauge group)

- Jeffrey-Kirwan residue integral

Elliptic genus for IIA brane model

[fields & quantum numbers]

	scalars	spinors	vectors
vector	$\cancel{\langle\varphi^{A\dot{A}}\rangle} = u^{A\dot{A}}$	$\cancel{\lambda_-^{\dot{I}A}}$, $\lambda_+^{\dot{I}\dot{A}}$	A_μ
adj. hyper	$Z^{I\dot{I}}$	$\Psi_-^{I\dot{A}}$, $\cancel{\Psi_+^{IA}}$	
fund. hyper	$q^{\dot{I}}$	$\psi_-^{\dot{A}}$, ψ_+^{A}	

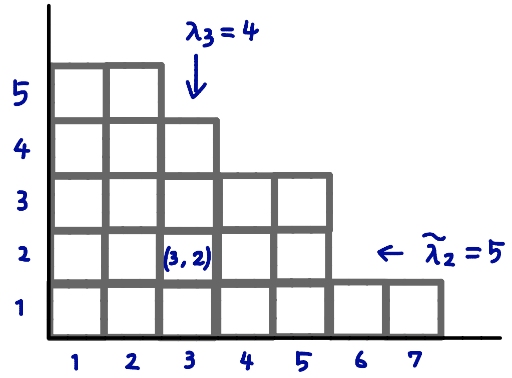
A quick way to evaluate it is to localize in Higgs branch

$$\mathcal{M} = \left\{ \begin{array}{r} q\bar{q} - \bar{\tilde{q}}\tilde{q} + [z,\bar{z}] + [\tilde{z},\bar{\tilde{z}}] + \zeta\cdot\mathbf{1}_{N\times N} = 0 \\ q\tilde{q} + [z,\tilde{z}] = 0 \end{array} \right\} \Big/ U(N)$$

$= $ (ADHM for U(1) N-instantons)

T^2 partition function of a SUSY sigma model on $\mathcal{M}$

with $(\text{SUSY})^2 =$ (a symmetry acting on $\mathcal{M}$ whose fixed points are labeled by Young diagrams with N boxes)



$$Z_N = \sum_{Y} \prod_{(i,j)\in Y} \frac{\theta_1\left(m + (i-\frac{1}{2})\epsilon_1 + (j-\frac{1}{2})\epsilon_2\right)\theta_1\left(m - (i-\frac{1}{2})\epsilon_1 - (j-\frac{1}{2})\epsilon_2\right)}{\theta_1\left((i-\tilde{\lambda}_j)\epsilon_1 + (\lambda_i - j - 1)\epsilon_2\right)\theta_1\left((\tilde{\lambda}_j - i - 1)\epsilon_1 + (j-\lambda_i)\epsilon_2\right)}$$

(agrees with topological vertex computation)

$$Z_N = \sum_{Y} \prod_{(i,j)\in Y} \frac{\theta_1\left(m+(i-\frac{1}{2})\epsilon_1+(j-\frac{1}{2})\epsilon_2\right)\theta_1\left(m-(i-\frac{1}{2})\epsilon_1-(j-\frac{1}{2})\epsilon_2\right)}{\theta_1\left((i-\tilde{\lambda}_j)\epsilon_1+(\lambda_i-j-1)\epsilon_2\right)\theta_1\left((\tilde{\lambda}_j-i-1)\epsilon_1+(j-\lambda_i)\epsilon_2\right)}$$

Remarks

When $\epsilon_1+\epsilon_2=\pm 2m$, Z_N all vanish because they all contain

$$Z_1 = \frac{\theta_1\left(m+\frac{1}{2}(\epsilon_1+\epsilon_2)\right)\theta_1\left(m-\frac{1}{2}(\epsilon_1+\epsilon_2)\right)}{\theta_1(\epsilon_1)\,\theta_1(\epsilon_2)}$$

This is when one can turn on u (mass for fund. matters).

For $u\to\infty$ one has $U(N)$ $\mathcal{N}=(4,4)$ SYM, of which the $U(1)$ part has vanishing elliptic genus.

$\Rightarrow$ $Z_N/Z_1 \xrightarrow{m=\frac{1}{2}(\epsilon_1+\epsilon_2)}$ Elliptic genus of $(4,4)$ $SU(N)$ SYM

Elliptic genus for ABJM slab

The theory for $u=0$

$$\mathcal{L}_{2D} = L \cdot \mathrm{Tr}\Big[\frac{k}{2\pi}\sigma \cdot F_{01} - D_\mu \bar{Z}^I D^\mu Z_I + i\Psi_-^{I\dot{A}}\gamma^\mu D_\mu \Psi_{-I\dot{A}} + i\Psi_+^{\dot{I}A}\gamma^\mu D_\mu \Psi_{+\dot{I}A} \Big]$$

For theories of topological YM-type,

JK-residue formula does not apply. We have instead

$$Z_N \sim \int \prod_{i=1}^{N} \frac{dw_i d\bar{w}_i}{\mathrm{Im}\,\tau} \cdot \prod_{i,j} \frac{\theta_1\left(w_i - w_j + m + \frac{1}{2}(\epsilon_1+\epsilon_2)\right)\theta_1\left(w_i - w_j + m - \frac{1}{2}(\epsilon_1+\epsilon_2)\right)}{\theta_1\left(w_i - w_j + \epsilon_1\right)\theta_1\left(w_i - w_j + \epsilon_2\right)}$$

$$Z_N \sim \int \prod_{i=1}^{N} \frac{dw_i d\bar{w}_i}{\mathrm{Im}\,\tau} \cdot \prod_{i,j} \frac{\theta_1\left(w_i - w_j + m + \frac{1}{2}(\epsilon_1+\epsilon_2)\right)\theta_1\left(w_i - w_j + m - \frac{1}{2}(\epsilon_1+\epsilon_2)\right)}{\theta_1(w_i - w_j + \epsilon_1)\,\theta_1(w_i - w_j + \epsilon_2)}$$

Remarks :

- this Z_N contains $(Z_1)^N$, so vanishes faster than Z_N of IIA brane model in the limit $m \to \pm\frac{1}{2}(\epsilon_1+\epsilon_2)$.
- when $m = \pm\frac{1}{2}(\epsilon_1+\epsilon_2)$,

  one can turn on u to move to $\mathcal{N}=(4,4)$ SYM,

  but there $Z_N = Z_1 \cdot Z_{SU(N)\ SYM}$.

  Turning on u is not a smooth deformation as far as the behavior of Z_{T^2} is concerned.

Conclusion

We proposed a description of self-dual strings using ABJM.

Compared the elliptic genus with the known resuts

$\longrightarrow$ disagreement for $u=0$.

* * *

... Need more careful & systematic study of ABJM

boundary condition? boundary DOF?

shouldn't we have translation symmetry for $k=1$?

ABJM-Nahm configurations? moduli space?