Donaldson-Thomas theory for Calabi-Yau four-folds

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(Master thesis under supervision of Conan Leung)
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Gauge Theories in Higher Dimensions
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Background

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$\rightsquigarrow H^*_{DT_3}(Y, E)$ s.t $\quad \chi\big(H^*_{DT_3}(Y, E)\big) =$ Donaldson-Thomas invariant

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Question: $M = X^4_{\mathbb{C}}$ (CY_4) ?

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with $*_4^2 = 1$

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$$*_4 : \Omega^{0,2}(X, EndE) \to \Omega^{0,2}(X, EndE)$$

with $*_4^2 = 1 \rightsquigarrow DT_4$-equation

$$\begin{cases} F_+^{0,2} = 0 \quad i.e. \quad F^{0,2} + *_4F^{0,2} = 0 \\ F \wedge \omega^3 = 0 \end{cases}$$

DT_4 moduli spaces and their virtual cycles

Definition

DT_4 **moduli space** $\mathcal{M}_c^{DT_4} \triangleq \{DT_4 - solutions\}/_{\cong} \subseteq \mathcal{B}$.

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Issue (2), i.e.

$$\mathcal{L} \triangleq det\big((\wedge^{top} Ext_+^2(E,E))^{-1} \otimes \wedge^{top} Ext^1(E,E)\big) \cong \mathcal{M}_c^{DT_4} \times \mathbb{R} \quad ?$$

DT_4 moduli spaces and their virtual cycles

Theorem (C-Leung)

Given X: compact simply connected CY_4 with $H_3(X,\mathbb{Z}) = H^3(X,\mathbb{Z}) = 0$, $U(r)$ bundle $E \to X$, then $\mathcal{L}$ over $\mathcal{B}$ is trivial.

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Above conditions hold for complete intersections in product of projective spaces

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Compactness issue, note:

$$F^{0,2} = 0 \Rightarrow F_{+}^{0,2} = 0, \ ch_2(E) \in H^{2,2}(X, \mathbb{C})$$

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Lemma (Lewis)

Converse is true. In particular, if every Gieseker semi-stable sheaf is a slope stable bundle i.e. $\overline{\mathcal{M}}_c^{shf} = \mathcal{M}_c^{bdl} \neq \emptyset$*, then* $\mathcal{M}_c^{DT_4}$ *is compact.*

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In this case, $\mathcal{M}_c^{DT_4} \cong \mathcal{M}_c^{bdl}$ as SETs.

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Theorem (C-Leung)

Assume $\overline{\mathcal{M}}_c^{shf} = \mathcal{M}_c^{bdl} \neq \emptyset$, $\mathcal{L}$ *is oriented. Then*

$$\exists \ [\mathcal{M}_c^{DT_4}]^{vir} \in H_r(\mathcal{B}, \mathbb{Z}).$$

The cycle is inv under deformation of complex str of X.

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The cycle is inv under deformation of complex str of X.

$r = 2 - \chi(X, EndE)$ is the virtual dim of $\mathcal{M}_c^{DT_4}$.

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Recall: analytic str of $\mathcal{M}_c^{bdl}$ is described by Kuranishi theory, i.e.

$$\exists\ \kappa : H^{0,1}(X, EndE) \to H^{0,2}(X, EndE),$$

s.t $\quad \mathcal{M}_c^{bdl} \cong \kappa^{-1}(0)$ locally.

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Claim: $\mathcal{M}_c^{DT_4} \cong \mathcal{M}_c^{bdl}$ as sets but **NOT** necessarily as real analytic spaces possibly with non-reduced structures.

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This is based on the following Kuranishi type thm for $\mathcal{M}_c^{DT_4}$

DT_4 moduli spaces and their virtual cycles

Theorem (C-Leung)

If $\mathcal{M}_c^{bdl} \neq \emptyset$, local Kuranishi model of $\mathcal{M}_c^{DT_4}$ at d_A is

$$\kappa_+ : H^{0,1}(X, EndE) \xrightarrow{\kappa} H^{0,2}(X, EndE) \xrightarrow{\pi_+} H_+^{0,2}(X, EndE),$$

where κ is a Kuranishi map for $\mathcal{M}_c^{bdl}$.

Furthermore, $\exists$ closed imbedding between analytic spaces possibly with non-reduced structures

$$\mathcal{M}_c^{bdl} \hookrightarrow \mathcal{M}_c^{DT_4}$$

which is also homeomorphism between topological spaces.

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In general, we hope to find an analytic space S and a homeomorphism

$$\overline{\mathcal{M}}_c^{shf} \rightarrow S$$

s.t. $S \cong \kappa_+^{-1}(0)$ locally at $\mathcal{F} \in \overline{\mathcal{M}}_c^{shf}$, where

$$\kappa_+ : Ext^1(\mathcal{F},\mathcal{F}) \rightarrow Ext^2_+(\mathcal{F},\mathcal{F}).$$

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We call such S the **generalized** DT_4 **moduli space** and denote it $\overline{\mathcal{M}}_c^{DT_4}$.

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In general, $\overline{\mathcal{M}}_c^{DT_4}$ may come from gluing local models.

DT_4 moduli spaces and their virtual cycles

Easiest case: If $\overline{\mathcal{M}}_c^{shf} = \mathcal{M}_c^{bdl} \neq \emptyset$, $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} = \mathcal{M}_c^{DT_4}$.

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If $\overline{\mathcal{M}}_c^{shf}$ is smooth, (i.e. all Kuranishi maps are zero), then $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} \cong \overline{\mathcal{M}}_c^{shf}$.

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Eg 2 (C-Leung)
If $X = K_Y$, with Y compact Fano 3-fold and $supp(\mathcal{F}) \subseteq Y$, then $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} \cong \overline{\mathcal{M}}_c^{shf}$.

DT_4 moduli spaces and their virtual cycles

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Proposition (C-Leung)

If $\forall \mathcal{F} \in \overline{\mathcal{M}}_c^{shf}$, $\exists$ $V_{\mathcal{F}}$ *s.t* $\left(Ext^2(\mathcal{F},\mathcal{F}), Q_{Serre}\right) \cong \left(T^*V_{\mathcal{F}}, Q_{std}\right)$ *and* $Image(\kappa_{\mathcal{F}}) \subseteq V_{\mathcal{F}}$, *where*

$$Q_{Serre} : Ext^2(\mathcal{F},\mathcal{F}) \otimes Ext^2(\mathcal{F},\mathcal{F}) \to Ext^4(\mathcal{F},\mathcal{F}) \cong \mathbb{C}$$

is the Serre duality pairing, Q_{std} *is the standard pairing between dual spaces, then* $\overline{\mathcal{M}}_c^{DT_4}$ *exists and* $\overline{\mathcal{M}}_c^{DT_4} \cong \overline{\mathcal{M}}_c^{shf}$.

DT_4 moduli spaces and their virtual cycles

In the above examples, $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} \cong \overline{\mathcal{M}}_c^{shf}$.

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What's more,

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This coincides with our earlier def of virtual cycles when semi-stable sheaves are stable bundles.

Axioms of DT_4 invariants

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$$\mu(\gamma, P) = P(c_1(\mathfrak{F}), c_2(\mathfrak{F}), ...,)/\gamma$$

where $\mathfrak{F}$ is the universal sheaf.

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Take $(\gamma, P) \rightsquigarrow DT_4$-inv$= \int_{[\overline{\mathcal{M}}_c^{DT_4}]^{vir}} \mu(\gamma, P)$.

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Since we only define DT_4-inv in several cases with different assumptions, to make all cases consistent, we propose several axioms that DT_4-invs should satisfy.

Axioms of DT_4 invariants

Axioms: Given a polarized CY_4 $\big(X, \mathcal{O}(1)\big)$, $c \in H^{even}(X, \mathbb{Q})$ and an orientation $o(\mathcal{L})$, the DT_4-*inv* is a map

$$DT_4(X, \mathcal{O}(1), c, o(\mathcal{L})) : Sym^*\big(H_*(X, \mathbb{Z}) \otimes \mathbb{Z}[x_1, x_2, ...]\big) \to \mathbb{Z},$$

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$$DT_4(X, \mathcal{O}(1), c, o(\mathcal{L})) = -DT_4(X, \mathcal{O}(1), c, -o(\mathcal{L}))$$

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(2) Deformation invariance

$$DT_4(X_0, \mathcal{O}(1)|_{X_0}, c, o(\mathcal{L}_0)) = DT_4(X_1, \mathcal{O}(1)|_{X_1}, c, o(\mathcal{L}_1))$$

$\big(X_t, \mathcal{O}(1)\big)$, $t \in [0, 1]$ deformation of cpx structures.

(3) Vanishing for negative virtual dimension

$$DT_4(X, \mathcal{O}(1), c, o(\mathcal{L})) = 0$$

if $2 - \chi(\mathcal{F}, \mathcal{F}) < 0$, where $\chi(\mathcal{F}, \mathcal{F})$ is determined by topology of X and c.

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(4) Vanishing for certain choice of c

$$DT_4(X, \mathcal{O}(1), c, o(\mathcal{L})) = 0,$$

if any one of the following two conditions is satisfied,
(i) $c|_{H^4(X,\mathbb{Q})}$ has no component in $H^{0,4}(X)$ and $c \notin \bigoplus_{i=0}^{4} H^{i,i}(X)$;
(ii) $c \in \bigoplus_{i=0}^{4} H^{i,i}(X)$, $\exists\ \varphi \in H^1(X, TX)$ such that $\varphi \lrcorner \left(c|_{H^{2,2}(X,\mathbb{Q})}\right) \neq 0$

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(5) Vanishing for compact hyper-Kähler manifolds

$$DT_4(X, \mathcal{O}(1), c, o(\mathcal{L})) = 0$$

if $Hol(X) = Sp(2)$.

(6) DT_4/DT_3 correspondence
For any compact Fano 3-fold $\big(Y, \mathcal{O}_Y(1)\big)$,

$$DT_4(K_Y, \pi^*\mathcal{O}_Y(1), c, o(\mathcal{O})) = DT_3(Y, \mathcal{O}_Y(1), c'),$$

$\pi : K_Y \to Y$ is projection, $c = (0, c|_{H^2_c(K_Y)} \neq 0, *, *, *)$.

In this setup, sheaves in $\overline{\mathcal{M}}_c^{shf}$ is of type $\iota_*(\mathcal{F})$, $\iota : Y \to K_Y$ the zero section and $c' = ch(\mathcal{F}) \in H^{even}(Y)$ uniquely determined by c. $o(\mathcal{O})$ denotes the natural complex orientation.

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(7) Normalizations

If virtual cycles exist (mentioned before)

$$DT_4\text{-inv} =< \mu(,), [\overline{\mathcal{M}}_c^{DT_4}]^{vir} >$$

Computational examples (DT_4/GW correspondence)

For smooth genus zero curve $C \hookrightarrow X$ with $\beta = [C] \in H_2(X, \mathbb{Z})$, $ch(\mathcal{I}_C) = (1, 0, 0, -PD(\beta), -1)$.

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Proposition (C-Leung)

Given compact CY_4: X, $c = (1, 0, 0, -PD(\beta), -1) \in H^{even}(X)$. Assume $\overline{\mathcal{M}}_c^{shf} = \{\mathcal{I}_C\} \cong \overline{\mathcal{M}}_{0,0}^{GW}(X, \beta)$ smooth, C: smooth imbedded $g = 0$ curve. Then $\mathcal{L}$ has natural orientation, $\overline{\mathcal{M}}_c^{DT_4}$ exists and $\overline{\mathcal{M}}_c^{DT_4} \cong \overline{\mathcal{M}}_{0,0}^{GW}(X, \beta)$. Furthermore,

(1) *if $Hol(X) = SU(4)$, then*

$$[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = [\overline{\mathcal{M}}_{0,0}^{GW}(X, \beta)]^{vir},$$

(2) *if $Hol(X) = Sp(2)$, then $[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = 0$ and*

$$[\overline{\mathcal{M}}_c^{DT_4}]^{vir}_{red} = [\overline{\mathcal{M}}_{0,0}^{GW}(X, \beta)]^{vir}_{red}.$$

Computational examples ($T^*\mathbb{P}^2$)

$X = T^*\mathbb{P}^2$, count sheaves w/ $supp(\mathcal{F}) \subseteq \mathbb{P}^2$ (scheme theoretically)

Computational examples ($T^*\mathbb{P}^2$)

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Proposition (C-Leung)

$$\iota_* : \overline{\mathcal{M}}_c^{shf}(\mathbb{P}^2) \xrightarrow{\cong} \overline{\mathcal{M}}_{c,\mathbb{P}^2}^{shf}(T^*\mathbb{P}^2), \quad \iota : \mathbb{P}^2 \to T^*\mathbb{P}^2$$

Then $\mathcal{L}$ *has natural orientation and* $[\overline{\mathcal{M}}_{c,\mathbb{P}^2}^{shf}(T^*\mathbb{P}^2)]^{vir} = 0$. *Furthermore,*

(1) *when* $rk(\mathcal{F}) \geq 2$, $[\overline{\mathcal{M}}_{c,\mathbb{P}^2}^{shf}(T^*\mathbb{P}^2)]^{vir}_{red} = 0$,

(2) *when* $rk(\mathcal{F}) = 1$,

$$[\overline{\mathcal{M}}_{c,\mathbb{P}^2}^{shf}(T^*\mathbb{P}^2)]^{vir}_{red} = \begin{cases} 1 & \text{if } c = (1,*,0) \\ \chi(Hilb^n(\mathbb{P}^2)) & \text{if } c = (1,0,-n) \end{cases}$$

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Eg. X generic smooth hyperplane section in $\mathbb{P}^1 \times \mathbb{P}^4$ of $(2,5)$ type

$$Chern\ class = [1 + (-1,1)|_X] \cdot [1 + (1,0)|_X],$$

Then $\overline{\mathcal{M}}_c^{shf}(L_r^X)$ (Gieseker moduli space *w.r.t* $L_r^X = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^4}(1,r)|_X$) is smooth and consists of slope-stable bdls only.

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(2) If $r = 1$, then $\overline{\mathcal{M}}_c^{DT_4} = \overline{\mathcal{M}}_c^{shf}(L_r^X) = \emptyset$, $\; [\overline{\mathcal{M}}_c^{DT_4}]^{vir} = 0$.

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Remark

Wall-crossing phenomenon exists in DT_4 theory

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Proposition (C-Leung)

X a generic smooth hyperplane section in $\mathbb{P}^1 \times \mathbb{P}^4$ of $(2,5)$ type

$$c = [1 + (-1,1)|_X] \cdot [1 + (\epsilon_1 + 1, \epsilon_2 - 1)|_X], \quad \epsilon_1, \epsilon_2 = 0, 1$$

$\overline{\mathcal{M}}_c^{shf}(L_r^X)$ *is the Gieseker moduli space,* $L_r^X = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^4}(1, r)|_X$

(1) *If* $\frac{15(2-\epsilon_2)}{6+5\epsilon_1+2\epsilon_2} < r < \frac{15(2-\epsilon_2)}{\epsilon_1(1+2\epsilon_2)}$, *then* $\overline{\mathcal{M}}_c^{DT_4} = \overline{\mathcal{M}}_c^{shf}(L_r^X) = \mathbb{P}^k$,

$$[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = [\mathbb{P}^k], \text{ where } k = (1 + \epsilon_1) \binom{6 - \epsilon_2}{4}.$$

(2) *If* $0 < r < \frac{15(2-\epsilon_2)}{6+5\epsilon_1+2\epsilon_2}$, *then* $\overline{\mathcal{M}}_c^{DT_4} = \overline{\mathcal{M}}_c^{shf}(L_r^X) = \emptyset$,

$$[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = 0.$$

Computational examples (ideal sheaves of one point)

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Proposition (C-Leung)

Let X be a compact CY_4, $c = (1,0,0,0,-1)$, then $\overline{\mathcal{M}}_c^{DT_4} \cong X$.

(1) If $Hol(X) = SU(4)$, then

$$[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = \pm PD\big(c_3(X)\big) \in H_2(X,\mathbb{Z}).$$

(2) If $Hol(X) = Sp(2)$, then

$$[\overline{\mathcal{M}}_c^{DT_4}]^{vir} = 0 \in H_1(X,\mathbb{Z}).$$

Furthermore, $[\overline{\mathcal{M}}_c^{DT_4}]^{vir}_{red} = 0 \in H_2(X,\mathbb{Z})$.

Some further directions

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We also define the equivariant DT_4-inv for ideal sheaves of curves $I_n(X, \beta)$ on any toric CY_4, X by virtual localization formula.

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The DT_4/GW correspondence in toric CY_4 cases would be interesting to study.

Relations with Borisov-Joyce's work

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A related work was done by Dennis Borisov and Dominic Joyce (see homepage of Borisov, preprint 2014). They used local 'Darboux charts' in the sense of Brav, Bussi and Joyce, the machinery of homotopical algebra and C^{∞}-algebraic geometry to get a compact derived C^{∞}-scheme with the same underlying topological structure as the Gieseker moduli space of stable sheaves.

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In our language, their results proved the existence of generalized DT_4 moduli spaces (C^∞-scheme version) in general. Furthermore, they defined the virtual fundamental class of the above derived C^∞-scheme.

Relations with Borisov-Joyce's work

In fact, BBJ's local 'Darboux theorem' mentioned above is important for their general gluing construction. We have a gauge theoretical proof of this 'Darboux theorem' for Gieseker moduli spaces of stable sheaves using gauge theory and Seidel-Thomas twists.

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In fact, BBJ's local 'Darboux theorem' mentioned above is important for their general gluing construction. We have a gauge theoretical proof of this 'Darboux theorem' for Gieseker moduli spaces of stable sheaves using gauge theory and Seidel-Thomas twists.

We then introduce a weaker condition on their local 'Darboux charts' to include local models induced from DT_4 equations. It turns out that the weaker condition is already sufficient for their gluing requirement which then indicates the equivalence of their virtual fundamental classes and DT_4 virtual cycles defined above.

Thank you for your attention !