Instantons on conical extensions of Sasakian manifolds

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Outline

Motivation

Cone constructions

- Link geometries in different dimensions
- Tool for constructing manifolds which rich structures [Bär 93; ...]

$\rightarrow$ Valuable in explicit model building

Instantons

- Have proven essential in the study of 4-manifolds [Donaldson 89], not exhaustively studied in higher dimensions
- Explicit solutions important for model building in physics [Harland, Nölle 11; Gemmer, Lechtenfeld 13; ...]

Geometrical detour – G-structures

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Reductions of principal bundles to subbundles

correspond to certain

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Ex.: Riemannian metric reduces $F(TM)$ to $SO(M,g)$

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Ex.: Riemannian metric reduces $F(TM)$ to $SO(M, g)$

G-structures

A **G-structure** on M is a reduction of the frame bundle $F(TM)$ to a principal subbundle $\mathcal{Q} \subset F(TM)$ with structure group $G \subset GL(D, \mathbb{R})$.

Geometrical detour – instanton conditions

Instanton conditions

Consider a **G-structure** $\mathcal{Q} \subset SO(M,g)$ on (M,g)

- $Ad(SO(M,g)) \simeq \Lambda^2 T^* M$
- **Restrict** this isomorphism:

  $Ad(\mathcal{Q}) \to \Lambda^2 T^* M$ defines **subbundle** $W(\mathcal{Q}) \subset \Lambda^2 T^* M$

  $\to$ **"Forms with components from $\mathfrak{g}$"**

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Instanton

An **instanton** on (M, g) w.r.t. $\mathcal{Q}$ is a connection on some gauge bundle $\mathfrak{B}$ over M whose field strength satisfies

$$F_A \in \Gamma\big(W(\mathcal{Q}) \otimes Ad(\mathfrak{B})\big) \subset \Omega^2\big(M, Ad(\mathfrak{B})\big)$$

Instanton conditions – implementations

Implementations of instanton conditions

- **Annihilate spinor** defining the G-structure

  $\gamma(F_A)(\epsilon) = 0$

  $\rightarrow$ **heterotic strings**

- **Eigenspace** of automorphisms of $\Lambda^2 T^* M$

  $*(Q \wedge F_A) = -F_A, \quad$ where $Q \in \Omega^{d-4}(M)$

  $\rightarrow$ **independent** of previous bundle constructions

  $\rightarrow SU(3)$ in $d = 6$: $W(\mathcal{Q}) \leftrightarrow *(\omega \wedge F_A) = -F_A$

Instanton constructions – the idea

Preparing the stage

Consider **manifold M^d with an H-structure**

- Induces **pushforward H-structure $\mathcal{Q}$ on $M^d \times I$**
- Apply transformations to $\mathcal{Q}$ to obtain new H-structure $\mathcal{Q}'$

Preparing the stage

Consider **manifold M^d with an H-structure**

- Induces **pushforward H-structure $\mathcal{Q}$ on $M^d \times I$**
- Apply transformations to $\mathcal{Q}$ to obtain new H-structure $\mathcal{Q}'$

$\Rightarrow$ In certain cases there exist **extensions $\mathcal{Q}' \subset \mathcal{P}$**,

where $\mathcal{P}$ is an G-structure on $M^d \times I$ with $H \subset G$

The ansatz

We consider $\mathcal{Q}' \subset \mathcal{P}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{m} \simeq T_x M^d$

Extend an $\mathfrak{h}$-valued instanton [Ivanova, Popov 12]: $\mathcal{A} = \Gamma + X_{\hat{\mu}} \otimes \beta^{\hat{\mu}}$
$\rightarrow X$ Ad-equivariant 1-form, β coframes on $M^d \times I$, $\hat{\mu} = (\mu, d+1)$

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Field strength: T = torsion of Γ

$$\mathcal{F} = R_\Gamma + \mathrm{d}X_{\hat{\mu}} \wedge \beta^{\hat{\mu}} + \frac{1}{2}\left([X_{\hat{\mu}}, X_{\hat{\nu}}] + T^{\hat{\sigma}}{}_{\hat{\mu}\hat{\nu}}\, X_{\hat{\sigma}}\right) \beta^{\hat{\mu}} \wedge \beta^{\hat{\nu}}$$
$$+ \Gamma^i \left([I_i, X_{\hat{\mu}}] - f_{i\hat{\mu}}{}^{\hat{\nu}}\, X_{\hat{\nu}}\right) \wedge \beta^{\hat{\mu}}$$

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$$+ \Gamma^i \left([I_i, X_{\hat{\mu}}] - f_{i\hat{\mu}}{}^{\hat{\nu}}\, X_{\hat{\nu}}\right) \wedge \beta^{\hat{\mu}}$$

X has **frame-independent coefficients** w.r.t. $\mathcal{Q}'$,

$$\Leftrightarrow \quad [I_i, X_{\hat{\mu}}] = \rho_*(I_i)^{\hat{\nu}}{}_{\hat{\mu}}\, X_{\hat{\nu}} = f_{i\hat{\mu}}{}^{\hat{\nu}}\, X_{\hat{\nu}}, \qquad \underbrace{\mathfrak{g}}_{I_A} = \underbrace{\mathfrak{h}}_{I_i} \oplus \underbrace{\mathfrak{m}}_{I_\mu}$$

The ansatz

With $X_{d+1} = 0$, $X_\mu = X_\mu(r)$ and $\dot{X}_\mu = \frac{\mathrm{d}}{\mathrm{d}r} X_\mu$:

$$\mathcal{F} = R_\Gamma + \left(\dot{X}_\mu + T^\delta{}_{d+1\,\mu}\, X_\delta\right) \beta^{d+1\,\mu} + \frac{1}{2}\left([X_\mu, X_\nu] + T^\sigma{}_{\mu\nu}\, X_\sigma\right) \beta^{\mu\nu}$$

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Adding a zero such that the second term becomes instanton

$$\mathcal{F} = R_\Gamma - \left(\dot{X}_\mu + T^\delta{}_{d+1\,\mu}\, X_\delta\right) \otimes \left(\beta^{\mu\, d+1} - \frac{1}{2} N^\mu{}_{\nu\kappa}\, \beta^{\nu\kappa}\right)$$

$$\underbrace{+\frac{1}{2}\left(T^\mu{}_{\nu\kappa}\, X_\mu + [X_\nu, X_\kappa] - N^\mu{}_{\nu\kappa}\left(\dot{X}_\mu + T^\delta{}_{d+1\,\mu}\, X_\delta\right)\right) \otimes \beta^{\nu\kappa}}_{\text{Require this to be an instanton}}$$

The ansatz

Matrix equations

Frame independence

$$[I_i, X_\mu] = f_{i\mu}{}^{\nu}\, X_\nu$$

Reduced instanton equations

$$[X_\mu, X_\nu] = -\,T^{\sigma}{}_{\mu\nu}\; X_\sigma + N^{\sigma}{}_{\mu\nu}\,\left(\dot{X}_\sigma + T^{\kappa}{}_{d+1\,\sigma}\; X_\kappa\right) + \mathcal{N}_{\mu\nu}$$

$$\mathcal{N} = \frac{1}{2}\, pr_{\mathfrak{h}}\Big(\big[X_\mu, X_\nu\big]\Big) \otimes \beta^\mu \wedge \beta^\nu$$

must be an instanton as well

Kähler-torsion sine-cones

Sasaki-Einstein manifolds

- $d = 2n + 1$, Structure group $\mathsf{SU}(n) \subset \mathsf{SO}(2n+1)$
- Geometric data

$$\eta = -\beta^{2n+1}, \qquad \omega^3 = \sum_{j=1}^{n} \beta^{2j-1} \wedge \beta^{2j},$$

$$P = \eta \wedge \omega^3, \qquad Q = \frac{1}{2}\,\omega^3 \wedge \omega^3,$$

$$\mathrm{d}\eta = 2\,\omega^3, \qquad \mathrm{d}P = 4\,Q$$

Sasaki-Einstein manifolds

Canonical SE connection [Harland, Nölle 11]

- Torsion $(a = 1, \ldots, 2n)$

$$T^a = \frac{n+1}{2n} P_{a\mu\nu}\,\beta^{\mu\nu}, \quad T^{2n+1} = P_{2n+1\,\mu\nu}\,\beta^{\mu\nu}$$

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- Is an **instanton** for the Sasaki-Einstein SU(n)-structure
- Its lift Γ^P to $M^{2n+1} \times I$ is an **instanton** for the pushforward SU(n)-structure $\mathcal{Q}$

Kähler-torsion sine-cones

Procedure [BILPS 14]

- Push SE SU(n)-structure forward to $M^{2n+1} \times (0, \Lambda\pi)$
- Transformation to the **sine-cone**:

  $\beta^\mu \mapsto \Lambda \sin(\frac{r}{\Lambda})\, \beta^\mu = \Lambda \sin(\varphi)\, \beta^\mu,\ \beta^{2n+1} \mapsto \beta^{2n+1}$

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- Push SE SU(n)-structure forward to $M^{2n+1} \times (0, \Lambda\pi)$
- Transformation to the **sine-cone**:
  $\beta^\mu \mapsto \Lambda \sin(\frac{r}{\Lambda})\, \beta^\mu = \Lambda \sin(\varphi)\, \beta^\mu,\ \beta^{2n+1} \mapsto \beta^{2n+1}$
- Extends to **Kähler-torsion** structure given by

$$g = \Lambda^2 \sin(\varphi)^2\, g_{2n+1} + \mathrm{d}r^2,$$

$$\omega = \Lambda^2\, \sin(\varphi)^2\, \omega^3 + \Lambda\, \sin(\varphi)\, \mathrm{d}r \wedge \eta$$

$\rightarrow$ approaches Calabi-Yau cone over M^{2n+1} as $\Lambda \to \infty$

Kähler-torsion sine-cones

Torsion of Bismut connection

$$\tilde{T} = J\,\mathrm{d}\omega = \frac{2}{\Lambda}\tan\left(\frac{\varphi}{2}\right)\omega \wedge \eta$$

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Sine-cone metric

$$g = \Lambda^2 \sin(\varphi)^2\; g_{2n+1} + \mathrm{d}r^2 = \Lambda^2 \sin(\varphi)^2 \left(g_{2n+1} + \mathrm{d}\tau(\varphi)^2\right)$$

$\Rightarrow$ conformal equivalence to the cylinder,

$$\tau(\varphi) = \log\left(2\Lambda \tan\left(\frac{\varphi}{2}\right)\right)$$

Kähler-torsion sine-cones

Reduction of the instanton equations $*Q_{\mathsf{KT}} \wedge F = - * F$

Ansatz: $\mathcal{A} = \Gamma + X_\mu \otimes \beta^\mu$, with Γ lift of SE canonical connection

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Reduction of the instanton equations $*Q_{\mathsf{KT}} \wedge F = - * F$

Ansatz: $\mathcal{A} = \Gamma + X_\mu \otimes \beta^\mu$, with Γ lift of SE canonical connection

With $\dot{X}_\mu = \frac{\mathrm{d}}{\mathrm{d}\tau} X_\mu$, $\underbrace{\mathfrak{su}(n+1)}_{I_A} = \underbrace{\mathfrak{su}(n)}_{I_i} \oplus \underbrace{\mathfrak{m}}_{I_\mu}$

Matrix equations for pullback to cylinder

$$[I_i, X_a] = f_{ia}^b X_b \quad \text{and} \quad [I_i, X_{2n+1}] = 0,$$

$$[X_a, X_b] = f_{ab}{}^{2n+1} \left(X_{2n+1} + \frac{1}{2n} \dot{X}_{2n+1} \right) + f_{ab}{}^j N_j(\tau),$$

$$[X_{2n+1}, X_a] = f_{2n+1\,a}{}^b \left(X_b + \frac{n}{n+1} \dot{X}_b \right)$$

Kähler-torsion sine-cones

Instanton eqns. on the cylinder

- Ansatz: $X_a = \psi \, I_a$, $X_{2n+1} = \chi \, I_{2n+1}$,
- Matrix equations reduce to (compare [Harland, Nölle 11])

$$\dot{\psi} = \frac{n+1}{n} \psi \, (\chi - 1) \quad \text{and} \quad \dot{\chi} = 2n \, (\psi^2 - \chi)$$

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$$\dot{\psi} = \frac{n+1}{n}\,\psi\,(\chi - 1) \quad \text{and} \quad \dot{\chi} = 2n\,(\psi^2 - \chi)$$

- **Solutions:**

$$\big(\psi(\tau),\, \chi(\tau)\big) = (0, 0),$$ **orignial instanton**

$$\big(\psi(\tau),\, \chi(\tau)\big) = (1, 1),$$ **∇^{LC} of CY metric cone**

$$\big(\psi(\tau),\, \chi(\tau)\big) = (0,\, C\, \exp(-4\,\tau))$$ **infinite action**

Conical hyper-Kähler-torsion structures

3-Sasakian manifolds

- $d = 4n + 3$, Structure group $\mathsf{Sp}(n) \subset \mathsf{SO}(4n+3)$
- Geometric data

$$\eta^\alpha = -\beta^{4n+\alpha}, \quad P = \frac{1}{3}\left(\sum_\alpha \eta^\alpha \wedge \omega^\alpha + \eta^{123}\right), \quad Q = \frac{1}{6}\sum_\alpha \omega^\alpha \wedge \omega^\alpha,$$

$$\omega^1 = \sum_{j=1}^{n}\left(\beta^{4j-3} \wedge \beta^{4j} + \beta^{4j-2} \wedge \beta^{4j-1}\right),$$

$$\omega^2 = \sum_{j=1}^{n}\left(-\beta^{4j-3} \wedge \beta^{4j-1} + \beta^{4j-2} \wedge \beta^{4j}\right),$$

$$\omega^3 = \sum_{j=1}^{n}\left(\beta^{4j-3} \wedge \beta^{4j-2} + \beta^{4j-1} \wedge \beta^{4j}\right)$$

3-Sasakian manifolds

These satisfy

$$\mathrm{d}\eta^{4n+\alpha} = \varepsilon^{\alpha}_{\beta\gamma}\,\eta^{\beta} \wedge \eta^{4n+\gamma} + 2\omega^{\alpha},$$
$$\mathrm{d}\omega^{\alpha} = 2\,\varepsilon^{\alpha}_{\beta\gamma}\,\eta^{4n+\beta} \wedge \omega^{\gamma}$$

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Canonical 3S connection Γ [Harland, Nölle 11]

- Torsion

$$T^{a} = \frac{3}{2} P_{a\mu\nu}\,\beta^{\mu\nu}, \quad T^{\alpha} = 3\,P_{4n+\alpha\,\mu\nu}\,\beta^{\mu\nu}$$

3-Sasakian manifolds

These satisfy

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$$\mathrm{d}\omega^{\alpha} = 2\, \varepsilon^{\alpha}_{\beta\gamma}\, \eta^{4n+\beta} \wedge \omega^{\gamma}$$

Canonical $3S$ connection Γ [Harland, Nölle 11]

- Torsion

$$T^{a} = \frac{3}{2} P_{a\mu\nu}\, \beta^{\mu\nu}, \quad T^{\alpha} = 3\, P_{4n+\alpha\,\mu\nu}\, \beta^{\mu\nu}$$

- Is an **instanton** for the 3-Sasakian Sp(n)-structure
- Its lift Γ^{P} to $M^{d} \times I$ is an **instanton** for the pushforward Sp(n)-structure $\mathcal{Q}$

Conical HKT structures

KT sine-cone: 3-Sasakian manifolds are SE, previous construction

HKT sine-cone [BILPS 14]: Need **three Kähler forms**

- On $M \times I$, introduce $\omega_\perp^\alpha \coloneqq \frac{1}{2}\eta^\alpha_{\rho\lambda}\, \beta^{4n+\rho} \wedge \beta^{4n+\lambda}$,

  $\alpha = 1,2,3,\ \rho,\lambda = 1,2,3,4,\ \eta^\alpha_{\rho\lambda}$ 't Hooft symbol

- For $f_1 = f_1(\varphi)$, $f_2 = f_2(\varphi)$, put

$$\tilde{\omega}^\alpha = \Lambda^2 \sin^2(\varphi)\,(f_1\,\omega^\alpha + f_2\,\omega_\perp^\alpha)$$

  $\rightarrow$ induce quaternionic structure on $M \times I$

HKT connection

- **Exists** $\Leftrightarrow$ $J^1\,\mathrm{d}\tilde{\omega}^1 = J^2\,\mathrm{d}\tilde{\omega}^2 = J^3\,\mathrm{d}\tilde{\omega}^3$ [Gauduchon 97]
- We have $J^1\,\mathrm{d}\tilde{\omega}^1 = J^2\,\mathrm{d}\tilde{\omega}^2 = J^3\,\mathrm{d}\tilde{\omega}^3 \sim P \Leftrightarrow$

$$\dot{f}\sin(\varphi) + 2f\cos(\varphi) = 0,$$

$$\dot{f}_2\sin(\varphi) + 2f_2\left(\cos(\varphi) - 1\right) + 2f = 0$$

for $f \coloneqq f_1 - f_2$

Conical HKT structures

Solved by

$$f_1 = \frac{2c_2}{\cos^4(\frac{\varphi}{2})} + \frac{2c}{\sin^2(\varphi)} \quad \text{and} \quad f_2 = \frac{2c_2}{\cos^4(\frac{\varphi}{2})} + \frac{c}{\sin^2(\varphi)}$$

Matches HK metric cone for $\Lambda \to \infty$ for $c = \frac{c1}{\Lambda^3}$

Metric on $M \times I$ reads

$$g = \Lambda^2 \sin(\varphi)^2 \left(f_1 \, \delta_{ab} \, \beta^a \otimes \beta^b + f_2 \, \delta_{\alpha\beta} \, \beta^\alpha \otimes \beta^\beta \right) + f_2 \, \mathrm{d}r^2$$

Conical HKT structures

Reduction of the instanton equations: Conf. Eq. to cylinder

- Instanton equation: $*Q_{\mathsf{HKT}} \wedge F = - * F$
- **Ansatz**: Extension of Γ, i.e. $\mathcal{A} = \Gamma + X_\mu(\tau) \otimes \beta^\mu$,

  where $\mu = 1, \ldots, 4n+3$,

  ${f_{AB}}^C$ structure constants of $\mathfrak{sp}(n+1) = \mathfrak{sp}(n) \oplus \mathfrak{m}$

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Reduced instanton equations for Γ on HKT space

$$[I_i, X_a] = {f_{ia}}^b X_b \quad \text{and} \quad [I_i, X_{4n+\alpha}] = 0,$$

$$[X_a, X_b] = {f_{ab}}^{4n+\alpha} X_{4n+\alpha} + {f_{ab}}^i N_i,$$

$$[X_a, X_{4n+\beta}] = {f_{a\,4n+\beta}}^b (X_b + \dot{X}_b),$$

$$[X_{4n+\alpha}, X_{4n+\beta}] = {f_{4n+\alpha\;4n+\beta}}^{4n+\gamma} (X_{4n+\gamma} + \frac{1}{2}\dot{X}_{4n+\gamma})$$

Conical HKT structures

Using $X_a(\tau) = \psi(\tau)\, I_a$ and $X_{4n+\alpha}(\tau) = \chi(\tau)\, I_{4n+\alpha}$, these reduce to

$\dot\psi = \psi\,(\chi - 1)$ **and** $\dot\chi = 2\,\chi\,(\chi - 1)$ **as well as** $\chi = \psi^2$

Solutions

$(\chi, \psi) = (0, 0)$	**canonical 3S connection**
$(\chi, \psi) = (1, \pm 1)$	**Stationary Extensions**
$\chi = \psi^2 = \frac{1}{2}\big(1 - \tanh(\tau - \tau_0)\big)$	**Interpolating solutions**

Nearly Kähler sine-cones – ansatz part I

SU(2)-structures in 5 dimensions

Defining sections [Conti, Salamon 05]

- 1-form $\eta = -\beta^5$,
- 2-forms $\omega^i = \frac{1}{2}\,\eta^i{}_{ab}\,\beta^a \wedge \beta^b$,

  $\eta^i{}_{ab}$ 't Hooft symbols, i.e.

  $$\omega^1 = \beta^{14} + \beta^{23}, \quad \omega^2 = -\beta^{13} + \beta^{24}, \quad \omega^3 = \beta^{12} + \beta^{34}$$

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Sasaki-Einstein SU(2)-structure

The SU(2)-structure $(\eta, \omega^i)_{i=1,2,3}$ is **Sasaki-Einstein** if

[Fernandez et al. 06]

$$\mathrm{d}\eta = 2\,\omega^3, \quad \mathrm{d}\omega^1 = -3\,\eta \wedge \omega^2, \quad \mathrm{d}\omega^2 = 3\,\eta \wedge \omega^1$$

Nearly Kähler sine-cones – ansatz part I

Procedure

- Push $\mathrm{SU}(2)$-structure to $M^5 \times I$
- Construct Kähler-torsion sine-cone over SE M^5
- Employ **additional deformation**:

$$\beta^\mu \mapsto \beta^\mu_\varphi := \exp\left(\frac{r}{2\,\Lambda}\,\eta^2\right)^\mu{}_\nu\,\beta^\nu$$

Yields global forms

$$\begin{aligned}
\tilde{\eta} &= \Lambda\,\sin(\varphi)\,\eta,\\
\tilde{\omega}^1 &= \Lambda^2\,\sin(\varphi)^2\,\big(\cos(\varphi)\,\omega^1 - \sin(\varphi)\,\omega^3\big),\\
\tilde{\omega}^2 &= \Lambda^2\,\sin(\varphi)^2\,\omega^2,\\
\tilde{\omega}^3 &= \Lambda^2\,\sin(\varphi)^2\,\big(\cos(\varphi)\,\omega^3 + \sin(\varphi)\,\omega^1\big)
\end{aligned}$$

Nearly Kähler sine-cones – ansatz part I

Nearly Kähler $SU(3)$-structure on $C_{\sin}(M^5)$:

$$\omega = \tilde{\omega}^3 + \mathrm{d}r \wedge \tilde{\eta}, \qquad \Omega^+ = \tilde{\omega}^2 \wedge \tilde{\eta} - \tilde{\omega}^1 \wedge \mathrm{d}r,$$

$$\Omega^- = -\left(\tilde{\omega}^1 \wedge \tilde{\eta} + \tilde{\omega}^2 \wedge \mathrm{d}r\right)$$

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Matrix equations for nK sine-cone for extensions of Γ^P

$$\left[I_i, X_\mu\right] = f_{i\mu}{}^{\nu}\, X_\nu,$$

$$\left[X_5, X_b\right] = \frac{3}{2\Lambda}\, \eta^3{}_b{}^a \left(\cot(\varphi)\, X_a + \frac{2}{3}\, \dot{X}_a\right) - \frac{3}{2\Lambda}\, \eta^1{}_b{}^a\, X_a,$$

$$\left[X_a, X_b\right] = \frac{2}{\Lambda}\, \eta^3{}_{ab} \left(\cot(\varphi)\, X_5 + \frac{1}{4}\, \dot{X}_5\right) - \frac{2}{\Lambda}\, \eta^1{}_{ab}\, X_5 + \mathcal{N}_{ab}$$

Here $\dot{X}_\mu = \frac{\mathrm{d}}{\mathrm{d}\varphi} X_\mu$, however, **no solutions** have been found

Nearly Kähler sine-cones – nK canonical connection

Nearly Kähler sine-cones – nK canonical connection

The nK canonical connection $\Gamma_{\mathfrak{su}(3)}$ of $C_{\sin}(M^5)$

- Can be computed from Maurer-Cartan equations

$$\mathrm{d}\beta^{\hat{\mu}} = -\Gamma^{\hat{\mu}}{}_{\hat{\nu}} \wedge \beta^{\hat{\nu}} + \frac{1}{2} T^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}}\, \beta^{\hat{\nu}} \wedge \beta^{\hat{\rho}}$$

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- $\mathfrak{su}(3)$-valued connection on $C_{\sin}(M^5)$,
  totally antisymmetric torsion, which is **parallel** w.r.t. $\Gamma_{\mathfrak{su}(3)}$

$\Rightarrow \Gamma_{\mathfrak{su}(3)}$ is an $\mathfrak{su}(3)$-valued instanton on $C_{\sin}(M^5)$

(e.g. [Harland, Nölle 11])

Nearly Kähler sine-cones – nK canonical connection

More on $\Gamma_{\mathfrak{su}(3)}$

- Splits into **new $\mathfrak{su}(2)$-valued connection** plus **$\mathfrak{m}$-valued rest**:

$$\Gamma_{\mathfrak{su}(3)} = \Gamma_{\mathfrak{su}(2)} + B(r)_\mu \otimes \beta^\mu$$

Nearly Kähler sine-cones – nK canonical connection

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Nearly Kähler sine-cones – ansatz part II

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- New ansatz: $\mathcal{A} = \Gamma_{\mathfrak{su}(2)} + X_\mu \otimes \beta^\mu$

Matrix equations for nK sine-cone using $\Gamma_{\mathfrak{su}(2)}$

$$
\begin{aligned}
\left[I_i, X_\mu\right] &= f_{i\mu}{}^{\nu}\, X_\nu, \\
\left[X_a, X_b\right] &= \frac{1}{2\Lambda}\, \eta^3{}_{ab} \left(5\cot(\varphi)\, X_5 + \dot{X}_5\right) - \frac{2}{\Lambda}\, \eta^1{}_{ab}\, X_5 + \mathcal{N}_{ab}, \\
\left[X_5, X_a\right] &= \frac{1}{2\Lambda}\, \eta^3{}_a{}^b \left(5\cot(\varphi)\, X_b + 2\,\dot{X}_b\right) \\
&\quad - \frac{3}{2\Lambda}\, \eta^1{}_a{}^b\, X_b + \frac{1}{2\Lambda}\, \eta^3{}_a{}^b\, \eta^{2c}{}_b\, X_c
\end{aligned}
$$

Nearly Kähler sine-cones – ansatz part II

- **Ansatz:** Recall $\Gamma_{\mathfrak{su}(3)} = \Gamma_{\mathfrak{su}(2)} + B(r)_\mu \otimes \beta^\mu$

  Thus, use $X_a = \psi(\varphi)\, B_a,\ X_5 = \chi(\varphi)\, B_5$

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  Thus, use $X_a = \psi(\varphi)\, B_a,\ X_5 = \chi(\varphi)\, B_5$

- **Algebraic equation:**

$$\sin(\varphi)\left(\psi^2(\varphi) - \chi(\varphi)\right) = 0$$

  Simplifies other equations to

$$\dot{\chi}(\varphi) = \dot{\psi}(\varphi) = 0 \quad \text{and} \quad \psi(\varphi)\big(\psi(\varphi)^2 - 1\big) = 0$$

Nearly Kähler sine-cones – ansatz part II

Solutions

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  Another instanton solution, reflection of β^a

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- Algebraic constraint excludes interpolating solutions

  for finite Λ

Half-flat cylinders

Half-flat cylinders

Procedure

- Lift SE $SU(2)$-structure to $M^5 \times I$
- Employ **deformation** depending on parameters (ζ, ϱ):

$$\begin{aligned} \beta_z^1 &= \beta^1, & \beta_z^2 &= \cos(\zeta)\,\beta^4 + \sin(\zeta)\,\beta^3, \\ \beta_z^3 &= \beta^2, & \beta_z^4 &= \cos(\zeta)\,\beta^3 - \sin(\zeta)\,\beta^4, \\ \beta_z^5 &= \varrho\,\beta^5 & \beta_z^6 &= \beta^6 \end{aligned}$$

$\Rightarrow$ **Two-parameter family of $SU(2)$-structures** on $M^5 \times I$

$\Rightarrow$ **Two-parameter family of half-flat $SU(3)$-structures**

Half-flat cylinders

Instanton condition

- Algebraic bundle condition $\Leftrightarrow \; *(\omega_z \wedge F_A) = -F_A$

$$\Leftrightarrow \; \omega_z \wedge \omega_z \wedge F_A = 0 \quad \text{and} \quad \Omega_z \wedge F_A = 0$$

- Here we **just impose**

$$\Omega_z \wedge F_A = 0$$

Implies $\mathrm{d}\Omega_z \wedge F_A = (W_1\,\omega_z \wedge \omega_z + W_2 \wedge \omega_z) \wedge F_A = 0$

- Note: $W_2^+ = \frac{3-4\varrho^2}{3\varrho} \left(\omega_z^3 + 2\,\eta_z \wedge dr\right)$

Half-flat cylinders

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Half-flat cylinders

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Reduction with $\mathcal{A} = \Gamma^P + X_\mu \otimes \beta^\mu$ leads to

Matrix equations for half-flat cylinders

$$\left[I_i, X_\mu\right] = f_{i\mu}{}^{\nu}\, X_\nu,$$

$$\left[X_a, X_b\right] = -2\,\varrho\,\eta^2{}_{ab}\, X_5 + \frac{2\varrho^2}{3}\,\eta^3{}_{ab}\,\dot{X}_5 + \mathcal{N}_{ab},$$

$$\left[X_5, X_a\right] = \frac{3}{2\varrho}\,\eta^2{}_a{}^b\, X_b + \eta^3{}_a{}^b\,\dot{X}_b$$

Half-flat cylinders

- **Ansatz introducing new constant parameter θ:**

$$X_1 = \psi\,(\cos(\theta)\,I_1 + \sin(\theta)\,I_4)\,,$$
$$X_4 = \psi\,(\cos(\theta)\,I_4 - \sin(\theta)\,I_1)\,,$$
$$X_2 = \psi\,(\cos(\theta)\,I_2 + \sin(\theta)\,I_3)\,,$$
$$X_3 = \psi\,(\cos(\theta)\,I_3 - \sin(\theta)\,I_2)\,,$$
$$X_5 = \chi\,I_5$$

- **Solutions: Two isolated instantons**

$$\theta = \frac{\pi}{4}\ : \qquad \psi = \pm 1\,, \quad \chi = -\frac{1}{\varrho},$$
$$\theta = \frac{3\pi}{4}: \qquad \psi = \pm 1\,, \quad \chi = +\frac{1}{\varrho}$$

Half-flat cylinders

New instantons are constant extensions of Γ^P

on constant deformations of Cartesian product $M \times I$

$\Rightarrow$ Lifts of instantons living on M^5

Thus instantons for either of the instanton conditions

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- Hopefully **interesting applications**, e.g. in **heterotic model building**

Thank you for your attention